Homework Set #9 Solutions

Exercises 2.9 (p. 174)

Assignment: Do #5, 7, 9, 12, 14, 16, 18, 20, 22, 24, 28

5. This is equivalent to determining if $\mathbf{w}$ is in the span of $\mathbf{v}_1$ and $\mathbf{v}_2$, so we form the augmented matrix and determine if the system is consistent:

$$\begin{bmatrix} 2 & -4 & 8 \\ 3 & -5 & 2 \\ -5 & 8 & -9 \end{bmatrix} \xrightarrow{R2 \to R2 - \frac{3}{2}R1} \begin{bmatrix} 2 & -4 & 8 \\ 0 & 1 & -10 \\ -5 & 8 & -9 \end{bmatrix}$$

$$\xrightarrow{R3 \to R3 + \frac{5}{2}R1} \begin{bmatrix} 2 & -4 & 8 \\ 0 & 1 & -10 \\ 0 & -2 & 11 \end{bmatrix} \xrightarrow{R3 \to R3 + 2R2} \begin{bmatrix} \boxed{2} & -4 & 8 \\ 0 & \boxed{1} & -10 \\ 0 & 0 & \boxed{-9} \end{bmatrix}$$

Since the system is inconsistent, $\mathbf{w}$ is not in the subspace.

7. (a) There are only three vectors in $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

(b) There are an infinite number of vectors in $\operatorname{Col} A$: it is the set of all linear combinations of A's three columns, and there are infinite number of those.

(c) To answer this, we must determine if $\mathbf{p}$ is in the span of the other vectors:

$$\begin{bmatrix} 2 & -3 & -4 & 6 \\ -8 & 8 & 6 & -10 \\ 6 & -7 & -7 & 11 \end{bmatrix} \xrightarrow{R2 \to R2 + 4R1} \begin{bmatrix} 2 & -3 & -4 & 6 \\ 0 & -4 & -10 & 14 \\ 6 & -7 & -7 & 11 \end{bmatrix}$$

$$\xrightarrow{R3 \to R3 - 3R1} \begin{bmatrix} 2 & -3 & -4 & 6 \\ 0 & -4 & -10 & 14 \\ 0 & 2 & 5 & -7 \end{bmatrix} \xrightarrow{R3 \to R3 + \frac{1}{2}R2} \begin{bmatrix} \boxed{2} & -3 & -4 & 6 \\ 0 & \boxed{-4} & -10 & 14 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since this system is consistent, $\mathbf{p}$ *is* in $\operatorname{Col} A$.

9. All we need to do is check the value of $A\mathbf{p}$ like so:

$$\begin{bmatrix} 2 & -3 & -4 \\ -8 & 8 & 6 \\ 6 & -7 & -7 \end{bmatrix} \begin{bmatrix} 6 \\ -10 \\ 11 \end{bmatrix} = \begin{bmatrix} -2 \\ -62 \\ 29 \end{bmatrix}$$

Obviously, $\mathbf{p}$ isn't a solution of $A\mathbf{x} = \mathbf{0}$, so it is not in $\operatorname{Nul} A$.

12. There are three unknowns, so solutions in $\operatorname{Nul} A$ are vectors in $\mathbb{R}^3$ ($p = 3$). Since there are 4 rows, $\operatorname{Col} A$ must be a subspace of $\mathbb{R}^4$ ($q = 4$).

14. Finding a nonzero vector in $\operatorname{Col} A$ is easy. For example, the first column $(1, 4, -5, 2)$ will work. Finding a nonzero vector in $\operatorname{Nul} A$ is harder. We have to solve the homogeneous equation:

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 4 & 5 & 7 & 0 \\ -5 & -1 & 0 & 0 \\ 2 & 7 & 11 & 0 \end{bmatrix} \xrightarrow{R2 \to R2 - 4R1} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -3 & -5 & 0 \\ -5 & -1 & 0 & 0 \\ 2 & 7 & 11 & 0 \end{bmatrix} \xrightarrow{R3 \to R3 + 5R1} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -3 & -5 & 0 \\ 0 & 9 & 15 & 0 \\ 2 & 7 & 11 & 0 \end{bmatrix}$$

$$\xrightarrow{R4 \to R4 - 2R1} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -3 & -5 & 0 \\ 0 & 9 & 15 & 0 \\ 0 & 3 & 5 & 0 \end{bmatrix} \xrightarrow{R3 \to R3 + 3R2} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -3 & -5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 5 & 0 \end{bmatrix}$$

$$\xrightarrow{R4 \to R4 + R2} \begin{bmatrix} \boxed{1} & 2 & 3 & 0 \\ 0 & -3 & -5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R2 \to -\frac{1}{3}R2} \begin{bmatrix} \boxed{1} & 2 & 3 & 0 \\ 0 & 1 & \frac{5}{3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R1 \to R1 - 2R2} \begin{bmatrix} 1 & 0 & -\frac{1}{3} & 0 \\ 0 & \boxed{1} & \frac{5}{3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This gives a general solution

$$\begin{cases} x_1 = \frac{1}{3}x_3 \\ x_2 = -\frac{5}{3}x_3 \\ x_3 \text{ free} \end{cases}$$

and a vector parametric solution

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3}x_3 \\ -\frac{5}{3}x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} \frac{1}{3} \\ -\frac{5}{3} \\ 1 \end{bmatrix}, \quad x_3 \text{ free}$$

Thus, any nonzero multiple of this vector will work, including $(1, -5, 3)$.

16. These vectors are scalar multiples of each other. Therefore, they are linearly dependent and cannot form a basis.

18. If we put these vectors into the columns of a matrix and reduce it to echelon form, we see that:

$$\begin{bmatrix} 1 & -5 & 7 \\ 1 & -1 & 0 \\ -2 & 2 & -5 \end{bmatrix} \xrightarrow{R2 \to R2 - R1} \begin{bmatrix} 1 & -5 & 7 \\ 0 & 4 & -7 \\ -2 & 2 & -5 \end{bmatrix}$$

$$\xrightarrow{R3 \to R3 + 2R1} \begin{bmatrix} 1 & -5 & 7 \\ 0 & 4 & -7 \\ 0 & -8 & 9 \end{bmatrix} \xrightarrow{R3 \to R3 + 2R2} \begin{bmatrix} \boxed{1} & -5 & 7 \\ 0 & \boxed{4} & -7 \\ 0 & 0 & \boxed{-5} \end{bmatrix}$$

Since this matrix has a pivot position in every row, it spans $\mathbb{R}^3$. Also, by the Invertible Matrix Theorem, its columns are linearly independent. Therefore, its columns form a basis for $\mathbb{R}^3$ by definition.

20. There are more vectors than entries in each vector. By Theorem 1.8, the vectors are linearly dependent. Therefore, they cannot form a basis.

22. From the echelon form, we see that the first and third columns of A are pivot columns. Therefore, those columns:

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -4 \end{bmatrix} \right\}$$

form a basis for $\operatorname{Col} A$. To find a basis for $\operatorname{Nul} A$, we must augment the matrix and reduce it to reduced echelon form:

$$\begin{bmatrix} \boxed{1} & -3 & 2 & 5 & 0 \\ 0 & 0 & 4 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R2 \to \frac{1}{4}R2} \begin{bmatrix} \boxed{1} & -3 & 2 & 5 & 0 \\ 0 & 0 & 1 & \frac{7}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R1 \to R1 - 2R2} \begin{bmatrix} 1 & -3 & 0 & \frac{3}{2} & 0 \\ 0 & 0 & \boxed{1} & \frac{7}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Uparrow \qquad\qquad \Uparrow \qquad\qquad \Uparrow$$

This gives a general solution

$$\begin{cases} x_1 = 3x_2 - \frac{3}{2}x_4 \\ x_3 = -\frac{7}{4}x_4 \qquad\qquad x_2,\, x_4 \text{ free} \end{cases}$$

and vector parametric form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -\frac{3}{2} \\ 0 \\ -\frac{7}{4} \\ 1 \end{bmatrix}, \qquad x_2,\, x_4 \text{ free}$$

These two vectors form a basis for $\operatorname{Nul} A$.

24. From the echelon form, we see that the first, second, and fourth columns of A are pivot columns. Therefore, those columns:

$$\left\{ \begin{bmatrix} 3 \\ -2 \\ 5 \\ -2 \end{bmatrix}, \begin{bmatrix} -5 \\ 4 \\ -9 \\ 6 \end{bmatrix}, \begin{bmatrix} 4 \\ 7 \\ -3 \\ 5 \end{bmatrix} \right\}$$

form a basis for $\operatorname{Col} A$. To find a basis for $\operatorname{Nul} A$, we continue the reduction:

$$\begin{bmatrix} \boxed{3} & -5 & -1 & 0 & 0 & 0 \\ 0 & 2 & 4 & 0 & 3 & 0 \\ 0 & 0 & 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R2 \to \frac{1}{2}R2} \begin{bmatrix} \boxed{3} & -5 & -1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Uparrow \qquad\qquad \Uparrow$$

$$\xrightarrow{R1\to R1+5R2} \begin{bmatrix} 3 & 0 & 9 & 0 & \frac{15}{2} & 0 \\ 0 & \boxed{1} & 2 & 0 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R1\to \frac{1}{3}R1} \begin{bmatrix} 1 & 0 & 3 & 0 & \frac{5}{2} & 0 \\ 0 & \boxed{1} & 2 & 0 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

giving general solution

$$\begin{cases} x_1 = -3x_3 - \frac{5}{2}x_5 \\ x_2 = -2x_3 - \frac{3}{2}x_5 \\ x_4 = -x_5 \end{cases}$$

and vector parametric form

$$\mathbf{x} = x_3 \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -\frac{5}{2} \\ -\frac{3}{2} \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad x_3,\ x_5 \text{ free}$$

These two vectors forms a basis for $\operatorname{Nul} A$.

28. No, $\operatorname{Col} A$ is not $\mathbb{R}^3$. Because A has four rows, $\operatorname{Col} A$ is a subspace of $\mathbb{R}^4$ and so doesn't have anything to do with $\mathbb{R}^3$. The dimension of $\operatorname{Nul} A$ is given by the rank theorem as $\dim \operatorname{Nul} A = 7 - \dim \operatorname{Col} A = 7 - 3 = 4$.