Homework Set #8 Solutions

Exercises 3.2 (p. 193)

Assignment: Do #6, 8, 12, 14, 23, 24, 26, 29, 30, 32, 34, 35, 36, 39, 40, 42

6. Reducing the matrix to echelon form:

$$\begin{bmatrix} \textcircled{1} & 5 & -3 \\ 3 & -3 & 3 \\ 2 & 13 & -7 \end{bmatrix} \xrightarrow{R2 \to R2 - 3R1} \begin{bmatrix} \textcircled{1} & 5 & -3 \\ 0 & -18 & 12 \\ 2 & 13 & -7 \end{bmatrix}$$

$$\xrightarrow{R3 \to R3 - 2R1} \begin{bmatrix} 1 & 5 & -3 \\ 0 & \textcircled{-18} & 12 \\ 0 & 3 & -1 \end{bmatrix} \xrightarrow{R3 \to R3 + \frac{1}{6}R2} \begin{bmatrix} \boxed{1} & 5 & -3 \\ 0 & \boxed{-18} & 12 \\ 0 & 0 & \boxed{1} \end{bmatrix}$$

We used only row replacement, so the determinant is the product of the elements on the main diagonal of the echelon form: $(1)(-18)(1) = -18$.

8. Reducing the matrix to echelon form:

$$\begin{bmatrix} \textcircled{1} & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 5 & 4 & -3 \\ -3 & -7 & -5 & 2 \end{bmatrix} \xrightarrow{R3 \to R3 - 2R1} \begin{bmatrix} \textcircled{1} & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & -1 & -2 & 5 \\ -3 & -7 & -5 & 2 \end{bmatrix} \xrightarrow{R4 \to R4 + 3R1} \begin{bmatrix} 1 & 3 & 3 & -4 \\ 0 & \textcircled{1} & 2 & -5 \\ 0 & -1 & -2 & 5 \\ 0 & 2 & 4 & -10 \end{bmatrix}$$

$$\xrightarrow{R3 \to R3 + R2} \begin{bmatrix} 1 & 3 & 3 & -4 \\ 0 & \textcircled{1} & 2 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & -10 \end{bmatrix} \xrightarrow{R4 \to R4 - 2R2} \begin{bmatrix} \boxed{1} & 3 & 3 & -4 \\ 0 & \boxed{1} & 2 & -5 \\ 0 & 0 & \boxed{0} & 0 \\ 0 & 0 & 0 & \boxed{0} \end{bmatrix}$$

Again, we used only row replacement, so the determinant is just the product of the elements on the main diagonal of the echelon form: $(1)(1)(0)(0) = 0$.

12. Cofactor expansion down the fourth column gives

$$\det A = -0 + 0 - 6\begin{vmatrix} -1 & 2 & 3 \\ 3 & 4 & 3 \\ 4 & 2 & 4 \end{vmatrix} + 3\begin{vmatrix} -1 & 2 & 3 \\ 3 & 4 & 3 \\ 5 & 4 & 6 \end{vmatrix}$$

Row reducing the first matrix to echelon form:

$$\begin{bmatrix} \textcircled{-1} & 2 & 3 \\ 3 & 4 & 3 \\ 4 & 2 & 4 \end{bmatrix} \xrightarrow{R2 \to R2+3R1} \begin{bmatrix} \textcircled{-1} & 2 & 3 \\ 0 & 10 & 12 \\ 4 & 2 & 4 \end{bmatrix} \xrightarrow{R3 \to R3+4R1} \begin{bmatrix} -1 & 2 & 3 \\ 0 & \textcircled{10} & 12 \\ 0 & 10 & 16 \end{bmatrix} \xrightarrow{R3 \to R3-R2} \begin{bmatrix} \boxed{-1} & 2 & 3 \\ 0 & \boxed{10} & 12 \\ 0 & 0 & \boxed{4} \end{bmatrix}$$

Since we used only row replacement, the determinant of the first matrix is $(-1)(10)(4) = -40$.

For the second matrix, reducing it to echelon form:

$$\begin{bmatrix} \textcircled{-1} & 2 & 3 \\ 3 & 4 & 3 \\ 5 & 4 & 6 \end{bmatrix} \xrightarrow{R2 \to R2+3R1} \begin{bmatrix} \textcircled{-1} & 2 & 3 \\ 0 & 10 & 12 \\ 5 & 4 & 6 \end{bmatrix} \xrightarrow{R3 \to R3+5R1} \begin{bmatrix} -1 & 2 & 3 \\ 0 & \textcircled{10} & 12 \\ 0 & 14 & 21 \end{bmatrix} \xrightarrow{R3 \to R3-\frac{7}{5}R2} \begin{bmatrix} \boxed{-1} & 2 & 3 \\ 0 & \boxed{10} & 12 \\ 0 & 0 & \boxed{\frac{21}{5}} \end{bmatrix}$$

That is, the determinant of the second matrix is $(-1)(10)(21/5) = -42$.

Therefore, the determinant of the original matrix is

$$\det A = -6(-40) + 3(-42) = 114$$

14. Cofactor expansion down the third column gives

$$\det A = 1\begin{vmatrix} 1 & 3 & -3 \\ -3 & 4 & 8 \\ 3 & -4 & 4 \end{vmatrix} - 0 + (-2)\begin{vmatrix} -3 & -2 & -4 \\ 1 & 3 & -3 \\ 3 & -4 & 4 \end{vmatrix} - 0$$

Row reducing the first matrix to echelon form:

$$\begin{bmatrix} \textcircled{1} & 3 & -3 \\ -3 & 4 & 8 \\ 3 & -4 & 4 \end{bmatrix} \xrightarrow{R2 \to R2+3R1} \begin{bmatrix} \textcircled{1} & 3 & -3 \\ 0 & 13 & -1 \\ 3 & -4 & 4 \end{bmatrix}$$

$$\xrightarrow{R3 \to R3-3R1} \begin{bmatrix} 1 & 3 & -3 \\ 0 & \textcircled{13} & -1 \\ 0 & -13 & 13 \end{bmatrix} \xrightarrow{R3 \to R3+R2} \begin{bmatrix} \boxed{1} & 3 & -3 \\ 0 & \boxed{13} & -1 \\ 0 & 0 & \boxed{12} \end{bmatrix}$$

We used only row replacement, so the determinant is $(1)(13)(12) = 156$.

Row reducing the second matrix to echelon form:

$$\begin{bmatrix} \textcircled{-3} & -2 & -4 \\ 1 & 3 & -3 \\ 3 & -4 & 4 \end{bmatrix} \xrightarrow{R2 \to R2+\frac{1}{3}R1} \begin{bmatrix} \textcircled{-3} & -2 & -4 \\ 0 & \frac{7}{3} & -\frac{13}{3} \\ 3 & -4 & 4 \end{bmatrix}$$

$$\xrightarrow{R3 \to R3+R1} \begin{bmatrix} -3 & -2 & -4 \\ 0 & \textcircled{\frac{7}{3}} & -\frac{13}{3} \\ 0 & -6 & 0 \end{bmatrix} \xrightarrow{R3 \to R3+\frac{18}{7}R2} \begin{bmatrix} \boxed{-3} & -2 & -4 \\ 0 & \boxed{\frac{7}{3}} & -\frac{13}{3} \\ 0 & 0 & \boxed{-\frac{78}{7}} \end{bmatrix}$$

Again, we only used row replacement giving a determinant of $(-3)(7/3)(-78/7) = 78$.

Therefore,

$$\det A = 1(156) - 2(78) = 0$$

23. Expanding across the first row:

$$\det A = 2\begin{vmatrix} -7 & -5 & 0 \\ 8 & 6 & 0 \\ 7 & 5 & 4 \end{vmatrix} - 0 + 0 - 8\begin{vmatrix} 1 & -7 & -5 \\ 3 & 8 & 6 \\ 0 & 7 & 5 \end{vmatrix}$$

For the first matrix, expanding down the third column

$$\begin{vmatrix} -7 & -5 & 0 \\ 8 & 6 & 0 \\ 7 & 5 & 4 \end{vmatrix} = 0 - 0 + 4\begin{vmatrix} 8 & 6 \\ 7 & 5 \end{vmatrix} = -8$$

For the second matrix, expanding down the first column

$$\begin{vmatrix} 1 & -7 & -5 \\ 3 & 8 & 6 \\ 0 & 7 & 5 \end{vmatrix} = 1\begin{vmatrix} 8 & 6 \\ 7 & 5 \end{vmatrix} - 3\begin{vmatrix} -7 & -5 \\ 7 & 5 \end{vmatrix} + 0 = -2$$

Therefore,

$$\det A = 2(-8) - 8(-2) = 0$$

so the determinant is singular (not invertible).

24. Forming the matrix with the given columns and expanding down the second column

$$\begin{vmatrix} 4 & -7 & -3 \\ 6 & 0 & -5 \\ -7 & 2 & 6 \end{vmatrix} = -(-7)\begin{vmatrix} 6 & -5 \\ -7 & 6 \end{vmatrix} + 0 - 2\begin{vmatrix} 4 & -3 \\ 6 & -5 \end{vmatrix} = 7(1) - 2(-2) = 11$$

Since the determinant is nonzero, the matrix is invertible, so by the Invertible Matrix Theorem, the original vectors are linearly independent.

26. Forming the matrix with the given columns and expanding down the fourth column

$$\begin{vmatrix} 3 & 2 & -2 & 0 \\ 5 & -6 & -1 & 0 \\ -6 & 0 & 3 & 0 \\ 4 & 7 & 0 & -3 \end{vmatrix} = -0 + 0 - 0 + (-3)\begin{vmatrix} 3 & 2 & -2 \\ 5 & -6 & -1 \\ -6 & 0 & 3 \end{vmatrix}$$

Expanding this matrix down the second column

$$\begin{vmatrix} 3 & 2 & -2 \\ 5 & -6 & -1 \\ -6 & 0 & 3 \end{vmatrix} = -2\begin{vmatrix} 5 & -1 \\ -6 & 3 \end{vmatrix} + (-6)\begin{vmatrix} 3 & -2 \\ -6 & 3 \end{vmatrix} - 0 = -2(9) + (-6)(-3) = 0$$

Thus, the determinant of the original matrix is 0, too. Therefore, by the Invertible Matrix Theorem, the original vectors are linearly dependent.

29. By the multiplicative property for determinants

$$\det B^5 = \det(BBBBB) = (\det B)^5$$

We can calculate $\det B$ by expanding down the second column:

$$\det B = -0 + 1 \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} - 2 \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = -2$$

Therefore, $\det B^5 = (-2)^5 = -32$.

30. Suppose rows i and j of A are equal. Then, the row replacement operation $Ri \to Ri - Rj$ produces a matrix B whose ith row is all zeros. The determinant of B can be calculated by a cofactor expansion along this all-zero row, giving $\det B = 0 + 0 + \cdots + 0 = 0$. However, by Theorem 3.3, we have $\det A = \det B$, so the determinant of A is zero, too.

The same is true for equal columns because column replacement doesn't change the determinant either.

32. We can form rA from A be scaling each row by r. A single scaling operation multiplies the determinant by r, so n scaling operations (one for each row) will multiply it by r a total of n times:

$$\det(rA) = \underbrace{r \dots r}_{n \text{ times}} \det(A) = r^n \det(A)$$

34. Because $PP^{-1} = I$, we have $(\det P)(\det P^{-1}) = \det I = 1$. Therefore,

$$\det(PAP^{-1}) = (\det P)(\det A)(\det P^{-1}) = (\det P)(\det P^{-1})(\det A) = \det A$$

35. Since $U^TU = I$, we have $(\det U^T)(\det U) = \det I = 1$. However, $\det U^T = \det U$. Therefore, $(\det U)^2 = 1$. The only possibilities are that $\det U = 1$ or $\det U = -1$.

36. Since $\det A^4 = \det(AAAA) = (\det A)^4$, if $\det A \neq 0$, we would have $\det A^4 \neq 0$. Therefore, as $\det A^4 = 0$, we must have $\det A = 0$.

39. (a) $\det AB = (\det A)(\det B) = 4(-3) = -12$

(b) From exercise 32, $\det 5A = 5^3 \det A = 125(4) = 500$

(c) $\det B^T = \det B = -3$

(d) $\det A^{-1} = \dfrac{1}{\det A} = \dfrac{1}{4}$

(e) $\det B^{-1}AB = (\det B^{-1})(\det A)(\det B) = \dfrac{1}{-3}(4)(-3) = 4$

40. (a) $\det AB = (\det A)(\det B) = (-1)(2) = -2$

(b) $\det B^5 = (\det B)^5 = 2^5 = 32$

(c) By exercise 32, $\det 2A = 2^4 \det A = 16(-1) = -16$

(d) $\det A^T A = (\det A^T)(\det A) = (\det A)(\det A) = (-1)(-1) = 1$

(e) $\det B^{-1}AB = (\det B^{-1})(\det A)(\det B) = \frac{1}{2}(-1)(2) = -1$

42. Note that $\det A + \det B = 1 + ad - bc$. Since

$$\det(A+B) = \begin{vmatrix} 1+a & b \\ c & 1+d \end{vmatrix} = (1+a)(1+d) - bc = 1 + a + d + ad - bc$$

we have $\det A + \det B = \det(A+B)$ iff $1 + ad - bc = 1 + a + d + ad - bc$ iff $0 = a + d$.

Exercises 3.3 (p. 204)

Assignment: Do #11, 17, 18, 20, 22, 26

11.

$$\operatorname{adj} A = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = \begin{bmatrix} +\begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} -2 & -1 \\ 1 & 1 \end{vmatrix} & +\begin{vmatrix} -2 & -1 \\ 0 & 0 \end{vmatrix} \\ -\begin{vmatrix} 3 & 0 \\ -1 & 1 \end{vmatrix} & +\begin{vmatrix} 0 & -1 \\ -1 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & -1 \\ 3 & 0 \end{vmatrix} \\ +\begin{vmatrix} 3 & 0 \\ -1 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & -2 \\ -1 & 1 \end{vmatrix} & +\begin{vmatrix} 0 & -2 \\ 3 & 0 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -3 & -1 & -3 \\ 3 & 2 & 6 \end{bmatrix}$$

Calculating the determinant by expansion across the second row

$$\det A = -3\begin{vmatrix} -2 & -1 \\ 1 & 1 \end{vmatrix} + 0 - 0 = 3$$

By the Inverse Formula,

$$A^{-1} = \frac{1}{\det A}\operatorname{adj} A = \frac{1}{3}\begin{bmatrix} 0 & 1 & 0 \\ -3 & -1 & -3 \\ 3 & 2 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 1/3 & 0 \\ -1 & -1/3 & -1 \\ 1 & 2/3 & 2 \end{bmatrix}$$

17. We showed this in class, but:

$$\operatorname{adj}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \end{bmatrix} = \begin{bmatrix} \det A_{11} & -\det A_{21} \\ -\det A_{12} & \det A_{22} \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

So,

$$A^{-1} = \frac{1}{\det A}\operatorname{adj} A = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

18. By the formula, each entry of A^{-1} is of the form $C_{ij}/\det A = C_{ij} = (-1)^{i+j} \det A_{ij}$. However, the determinant of a matrix with integer entries is an integer. (This can be proved by induction on the size of the matrix: it's true for 1×1 matrices, and if it's true for a $n \times n$ matrices then a cofactor expansion shows it's true for $(n+1) \times (n+1)$ matrices. Therefore, the entries of A^{-1} are integers.

20. Since $(3, -2) = (-1, 3) + (4, -5)$, this is the parallelogram determined by the columns of the matrix

$$A = \begin{bmatrix} -1 & 4 \\ 3 & -5 \end{bmatrix}$$

(Draw a picture if you can't see why these are the right columns to use.) By Theorem 3.9, the area is $|\det A| = |5 - 12| = 7$.

22. If we shift the first point to the origin (by adding $(0, 2)$ to each point), we see that the original parallelogram has the same area as the parallelogram with points $(0, 0)$, $(6, 1)$, $(-3, 3)$, and $(3, 4)$. Since $(6, 1) + (-3, 3) = (3, 4)$, this new parallelogram is the one determined by the columns of the matrix

$$A = \begin{bmatrix} 6 & -3 \\ 1 & 3 \end{bmatrix}$$

(Again, a picture may help if you can't see why these are the right columns.) By Theorem 3.9, the area is $|\det A| = |18 + 3| = 21$.

26. Each point in $\mathbf{p} + S$ is a point $\mathbf{p} + \mathbf{s}$ for some $\mathbf{s} \in S$. This has image

$$T(\mathbf{p} + \mathbf{s}) = T(\mathbf{p}) + T(\mathbf{s})$$

which is a point $T(\mathbf{s})$ of the transformed set $T(S)$ translated by $T(\mathbf{p})$. That is, every point in the image of $\mathbf{p} + S$ is a point in $T(\mathbf{p}) + T(S)$.

On the other hand, every point in $T(\mathbf{p}) + T(S)$ can be written $T(\mathbf{p}) + T(\mathbf{s})$ for some $\mathbf{s} \in S$. But, this is the image of $\mathbf{p} + \mathbf{s}$, which is a point in the translated set $\mathbf{p} + \mathbf{S}$. That is, every point in $T(\mathbf{p}) + T(S)$ is in the image of $\mathbf{p} + S$.

Exercises 2.9 (p. 174)

Assignment: Do #37, 38, 1, 2, 3, 4, 8, 10

37. (a) False. Part (ii) should read "if $\mathbf{u}$ and $\mathbf{v}$ are in H, then $\mathbf{u} + \mathbf{v}$ is in H, and part (iii) should read "if $\mathbf{u}$ is in H and c is a scalar, then $c\mathbf{u}$ is in H.

(b) True. (p. 166)

(c) True. (p. 167)

(d) False. The pivot columns of A form a basis for $\operatorname{Col} A$. The pivot columns of B generally won't work.

(e) True. By the Invertible Matrix Theorem, the columns of an invertible matrix span $\mathbb{R}^n$ and are linearly independent. By definition, these vectors form a basis for $\mathbb{R}^n$.

38. (a) False. That's not enough. We need H to be closed under addition and scalar multiplication as well.

(b) False. The column space is the span of the columns.

(c) False. It will be a subspace of $\mathbb{R}^m$.

(d) True. (p. 171)

(e) True. (p. 172)

1. Note that $(1, 0)$ is in H, but $-1(1, 0) = (-1, 0)$ is not. Therefore, H is not closed under scalar multiplication.

2. Note that $(0, 1)$ and $(-1, 0)$ are in H. But their sum is $(-1, 1)$, and that's not in H, so H is not closed under vector addition.

3. The vector $(x, 0)$ will be in H for a sufficiently small positive real number x. However, for a big scalar c, $c(x, 0) = (cx, 0)$ will fall outside H. For example, of the right boundary of H crosses the x-axis at $(.5, 0)$, then $(.25, 0)$ is in H, but $4(.25, 0) = (1, 0)$ is not.

4. The vector $(1, 1)$ is in H, but $-(1, 1) = (-1, -1)$ is not.

8. We need to determine if $\mathbf{p}$ is in the span of the columns of A. Therefore, form the augmented matrix $\begin{bmatrix} A & \mathbf{p} \end{bmatrix}$ and check the consistency of the system:

10. All we need to do is check if $A\mathbf{u}$ equals $\mathbf{0}$. In fact,

$$A\mathbf{u} = \begin{bmatrix} -3 & -2 & 0 \\ 0 & 2 & -6 \\ 6 & 3 & 3 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Therefore, $\mathbf{u}$ *is* in $\operatorname{Nul} A$.