## Homework Set \#7 Solutions

## Exercises 2.2 (p. 117)

Assignment: Do \#14, 17, 19, 35, 37
14. Multiply both sides of the equation by $A^{-1}$ on the right:

$$
\begin{aligned}
(B-C) A & =0 \\
(B-C) A A^{-1} & =0 A^{-1} \\
B-C & =0
\end{aligned}
$$

Therefore, $B=C$.
17. Multiply both sides of the equation by $B^{-1}$ on the right:

$$
\begin{aligned}
A B & =B C \\
A B B^{-1} & =B C B^{-1} \\
A & =B C B^{-1}
\end{aligned}
$$

19. We want to cancel the $C^{-1}$ and $B^{-1}$. Therefore, multiply on the left by $C$ and on the right by $B$ :

$$
\begin{aligned}
C^{-1}(A+X) B^{-1} & =I \\
C C^{-1}(A+X) B^{-1} & =C I \\
(A+X) B^{-1} & =C \\
(A+X) B^{-1} B & =C B \\
A+X & =C B \\
X & =C B-A
\end{aligned}
$$

35. This is an application of the technique mentioned in "Another View of Matrix Inversion" on page 116. To calculate only the third column of $A^{-1}$, we need only solve the equation $A \mathbf{x}=\mathbf{e}_{3}$. Reducing the augmented matrix:


$$
\begin{aligned}
& \xrightarrow{R 3 \rightarrow R 3+R 1}\left[\begin{array}{cccc}
-1 & -5 & -7 & 0 \\
0 & -5 & -8 & 0 \\
0 & -2 & -3 & 1
\end{array}\right] \xrightarrow{R 3 \rightarrow R 3-\frac{2}{5} R 2}\left[\begin{array}{cccc}
\begin{array}{|cc|}
\hline-1 & -5 \\
0 & -7 \\
0 & \boxed{-5}
\end{array} & -8 & 0 \\
0 & 0 & \left(\frac{1}{5}\right. & 1
\end{array}\right] \\
& \xrightarrow{R 3 \rightarrow 5 R 3}\left[\begin{array}{cccc}
\begin{array}{|ccc}
-1 & -5 & -7 \\
0
\end{array} \\
0 & \boxed{-5} & -8 & 0 \\
0 & 0 & 1 & 5 \\
\uparrow & \xrightarrow{R 1 \rightarrow R 1+7 R 3}\left[\begin{array}{cccc}
\boxed{-1} & -5 & 0 & 35 \\
0 & \boxed{-5} & -8 & 0 \\
0 & 0 & (1) & 5
\end{array}\right] \\
& \uparrow & \\
& &
\end{array}\right. \\
& \xrightarrow{R 2 \rightarrow R 2+8 R 3}\left[\begin{array}{cccc}
\begin{array}{|ccc}
-1 & -5 & 0 \\
\hline
\end{array} & 35 \\
0 & -5 & 0 & 40 \\
0 & 0 & \boxed{1} & 5
\end{array}\right] \xrightarrow{R 2 \rightarrow-\frac{1}{5} R 2}\left[\begin{array}{cccc}
\boxed{-1} & -5 & 0 & 35 \\
0 & (1) & 0 & -8 \\
0 & 0 & \boxed{1} & 5
\end{array}\right] \\
& \xrightarrow{R 1 \rightarrow R 1+5 R 2}\left[\begin{array}{cccc}
-1 & 0 & 0 & -5 \\
0 & \boxed{1} & 0 & -8 \\
0 & 0 & \boxed{1} & 5
\end{array}\right] \xrightarrow{R 1 \rightarrow-R 1}\left[\begin{array}{cccc}
1 & 0 & 0 & 5 \\
\uparrow & & 1 & 0 \\
\hline
\end{array}\right]
\end{aligned}
$$

Therefore, the column vector $(5,-8,5)$ is the third row the inverse.
37. Even though $C A=I_{2}, A$ isn't invertible. Only square matrices can be invertible, by definition.

## Exercises 2.3 (p. 123)

Assignment: Do \#1, 3, 5, 7, 9, 16, 19, 23

1. The determinant is $(-4)(-9)-16(3)=36-48=-12$, so the matrix is invertible.
2. The columns of the matrix are linearly dependent (since the second column is all zeros). By the Invertible Matrix Theorem (IMT), the matrix is not invertible.
3. Applying the row operations $R 1 \leftrightarrow R 3, R 3 \rightarrow R 3-5 R 1$, and $R 3 \rightarrow R 3+3 R 2$ we get an echelon form

$$
\left[\begin{array}{lll}
1 & 0 & 3 \\
0 & 3 & 4 \\
0 & 0 & 0
\end{array}\right]
$$

which has only two pivot positions. By the IMT, the original matrix is not invertible.
7. Applying the row operations $R 3 \rightarrow R 3+2 R 1, R 4 \rightarrow R 4-3 R 1$, and $R 4 \rightarrow R 4+4 R 2$, we get the echelon form

$$
\left[\begin{array}{cccc}
1 & 3 & 0 & -1 \\
0 & 1 & -2 & -1 \\
0 & 0 & 3 & 0 \\
0 & 0 & 0 & -4
\end{array}\right]
$$

which has four pivot positions. By the IMT, the original matrix is invertible.
9. The transpose of this matrix is in echelon form and has four pivot positions. By the IMT, this transpose must be invertible, and by another application of the IMT, if the transpose is invertible then the original matrix must be invertible.
16. $A \mathbf{x}=\mathbf{0}$ always has the trivial solution. This tells us nothing about whether or not the columns of $A$ span $\mathbb{R}^{n}$.
19. By the IMT, if the columns of $P$ are linearly independent, then $P$ is invertible. By Theorem $2.5, P \mathbf{x}=\mathbf{b}$ has a unique solution for every $\mathbf{b}$.
23. If $A$ is invertible, it has inverse $A^{-1}$. But, then, note that

$$
A^{2}\left(A^{-1} A^{-1}\right)=A\left(A A^{-1}\right) A^{-1}=A I A^{-1}=A A^{-1}=I
$$

Similarly, $\left(A^{-1} A^{-1}\right) A^{2}=I$, so $A^{2}$ has an inverse $A^{-1} A^{-1}$.

## Exercises 3.1 (p. 185)

Assignment: Do \#39, 40, 2, 4, 10, 12, 14, 19, 20, 22, 24
39. (a) True. (p. 181)
(b) False. The cofactor $C_{i j}$ is a number, not a matrix. It's the determinant of the matrix $A_{i j}$ times $(-1)^{i+j}$.
40. (a) False. The cofactor expansion across any row is equal to the cofactor expansion down any column.
(b) False. The determinant is the product of those entries.
2. Across the first row,

$$
\left|\begin{array}{rrr}
0 & 5 & 1 \\
4 & -3 & 0 \\
2 & 4 & 1
\end{array}\right|=0-5\left|\begin{array}{ll}
4 & 0 \\
2 & 1
\end{array}\right|+1\left|\begin{array}{rr}
4 & -3 \\
2 & 4
\end{array}\right|=0-5(4)+1(16+6)=2
$$

Down the second column,

$$
\left|\begin{array}{rrr}
0 & 5 & 1 \\
4 & -3 & 0 \\
2 & 4 & 1
\end{array}\right|=-5\left|\begin{array}{ll}
4 & 0 \\
2 & 1
\end{array}\right|+(-3)\left|\begin{array}{ll}
0 & 1 \\
2 & 1
\end{array}\right|-4\left|\begin{array}{ll}
0 & 1 \\
4 & 0
\end{array}\right|=-5(4)+(-3)(-2)-4(-4)=2
$$

4. Across the first row,

$$
\left|\begin{array}{lll}
1 & 3 & 5 \\
2 & 1 & 1 \\
3 & 4 & 2
\end{array}\right|=1\left|\begin{array}{ll}
1 & 1 \\
4 & 2
\end{array}\right|-3\left|\begin{array}{ll}
2 & 1 \\
3 & 2
\end{array}\right|+5\left|\begin{array}{ll}
2 & 1 \\
3 & 4
\end{array}\right|=1(2-4)-3(4-3)+5(8-3)=20
$$

Down the second column,

$$
\left|\begin{array}{lll}
1 & 3 & 5 \\
2 & 1 & 1 \\
3 & 4 & 2
\end{array}\right|=-3\left|\begin{array}{ll}
2 & 1 \\
3 & 2
\end{array}\right|+1\left|\begin{array}{cc}
1 & 5 \\
3 & 2
\end{array}\right|-4\left|\begin{array}{ll}
1 & 5 \\
2 & 1
\end{array}\right|=-3(4-3)+1(2-15)-4(1-10)=20
$$

10. Expanding across the second row,

$$
\operatorname{det} A=-0+0-3\left|\begin{array}{rrr}
1 & -2 & 2 \\
2 & -6 & 5 \\
5 & 0 & 4
\end{array}\right|+0
$$

Expanding the submatrix down the second column,

$$
\left|\begin{array}{rrr}
1 & -2 & 2 \\
2 & -6 & 5 \\
5 & 0 & 4
\end{array}\right|=-(-2)\left|\begin{array}{ll}
2 & 5 \\
5 & 4
\end{array}\right|+(-6)\left|\begin{array}{ll}
1 & 2 \\
5 & 4
\end{array}\right|-0=2(8-25)-6(4-10)=2
$$

Therefore, $\operatorname{det} A=-3(2)=-6$.
12. Since this is a (lower) triangular matrix, the determinant is just the product of the entries on the main diagonal $(4)(-1)(3)(-3)=36$. If you went to the trouble of writing out the cofactor expansion (always using the first row), you would get exactly the same product of diagonal entries.
14. Expanding down the fifth column,

$$
\operatorname{det} A=0-0+1\left|\begin{array}{rrrr}
6 & 3 & 2 & 4 \\
9 & 0 & -4 & 1 \\
3 & 0 & 0 & 0 \\
4 & 2 & 3 & 2
\end{array}\right|-0+0
$$

Expanding the submatrix across the third row,

$$
\left|\begin{array}{rrrr}
6 & 3 & 2 & 4 \\
9 & 0 & -4 & 1 \\
3 & 0 & 0 & 0 \\
4 & 2 & 3 & 2
\end{array}\right|=3\left|\begin{array}{rrr}
3 & 2 & 4 \\
0 & -4 & 1 \\
2 & 3 & 2
\end{array}\right|-0+0-0
$$

Expanding the submatrix down the first column,

$$
\left|\begin{array}{rrr}
3 & 2 & 4 \\
0 & -4 & 1 \\
2 & 3 & 2
\end{array}\right|=3\left|\begin{array}{rr}
-4 & 1 \\
3 & 2
\end{array}\right|-0+2\left|\begin{array}{rr}
2 & 4 \\
-4 & 1
\end{array}\right|=3(-8-3)+2(2+16)=3
$$

Therefore, $\operatorname{det} A=1(3(3))=9$.
19. The row operation is $R 1 \leftrightarrow R 2$, and the determinant changes from $a d-b c$ to $c b-d a=$ $-(a d-b c)$. That is, it changes sign.
20. The row operation is $R 2 \rightarrow k R 2$, and the determinant changes from $a d-b c$ to $a k d-b k c=$ $k(a d-b c)$. That is, it's multiplied by $k$.
22. The row operation is $R 1 \rightarrow R 1+k R 2$, and the determinant changes from $a d-b c$ to ( $a+$ $k c) d-(b+k d) c=a d+k c d-b c-k c d=a d-b c$. That is, it doesn't change.
24. The row operation is $R 1 \leftrightarrow R 2$. The determinant changes from (expanding along the first row):

$$
\begin{aligned}
\operatorname{det} A=a\left|\begin{array}{ll}
2 & 2 \\
5 & 6
\end{array}\right| & -b\left|\begin{array}{ll}
3 & 2 \\
6 & 6
\end{array}\right|+c\left|\begin{array}{ll}
3 & 2 \\
6 & 5
\end{array}\right| \\
& =a(12-10)-b(18-12)+c(15-12)=2 a-6 b+3 c
\end{aligned}
$$

to (again expanding along the first row):

$$
\begin{aligned}
\operatorname{det} B=3\left|\begin{array}{ll}
b & c \\
5 & 6
\end{array}\right|-2\left|\begin{array}{ll}
a & c \\
6 & 6
\end{array}\right|+2\left|\begin{array}{ll}
a & b \\
6 & 5
\end{array}\right| & =3(6 b-5 c)-2(6 a-6 c)+2(5 a-6 b) \\
& =18 b-15 c-12 a+12 c+10 a-12 b=-2 a+6 b-3 c
\end{aligned}
$$

Again, it changes the determinant's sign.

## Exercises 3.2 (p. 193)

Assignment: Do \#27, 28
27. (a) True. (p. 187)
(b) False. The boxed formula on p. 188 only works when the echelon form $U$ has been created without row scaling operations.
(c) True. By the Invertible Matrix Theorem, the matrix is not invertible, so by Theorem 3.4, its determinant is zero.
(d) False. (see warning on p. 191)
28. (a) True. The first changes the sign, and the second changes it back.
(b) False. This is only true if $A$ is a triangular matrix.
(c) False. The matrix $\left[\begin{array}{ll}1 & 2 \\ 3 & 6\end{array}\right]$ has determinant zero, but it has different rows, different columns, and no zeros.
(d) False. $\operatorname{det} A^{T}=\operatorname{det} A$.

