Homework Set #6 Solutions

Exercises 2.1 (p. 107)

Assignment: Do #17, 19, 21, 25

17. The size of A and AB tells us that the size of B must be 2×3. Furthermore, if $B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{bmatrix}$ then $AB = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & A\mathbf{b}_3 \end{bmatrix}$.

In particular, the first column of AB gives us the equation:

$$A\mathbf{b}_1 = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} \mathbf{b}_1 = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$$

The augmented matrix is

$$\begin{bmatrix} 1 & -2 & -1 \\ -2 & 5 & 6 \end{bmatrix}$$

which, when reduced, gives the unique solution

$$\mathbf{b}_1 = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$$

Similarly, the equation

$$A\mathbf{b}_2 = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} \mathbf{b}_2 = \begin{bmatrix} 2 \\ -9 \end{bmatrix}$$

has the unique solution

$$\mathbf{b}_2 = \begin{bmatrix} -8 \\ -5 \end{bmatrix}$$

19. The third column of AB is $A\mathbf{b}_3 = A(\mathbf{b}_1 + \mathbf{b}_2) = A\mathbf{b}_1 + A\mathbf{b}_2$. Therefore, the third column of AB is the sum of the first two columns of AB.

21. If B is $n \times p$, then the last column of AB is $A\mathbf{b}_p = \mathbf{0}$. If B has no column of all zeros, then $\mathbf{b}_p$ isn't all zeros. Therefore, $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution (namely $\mathbf{b}_p$), so the columns of A are linearly dependent.

25.

$$\mathbf{u}^T\mathbf{v} = \begin{bmatrix} -2 & 3 & -4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = -2a + 3b - 4c$$

$$\mathbf{v}^T\mathbf{u} = \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} -2 \\ 3 \\ -4 \end{bmatrix} = -2a + 3b - 4c$$

$$\mathbf{u}\mathbf{v}^T = \begin{bmatrix} -2 \\ 3 \\ -4 \end{bmatrix} \begin{bmatrix} a & b & c \end{bmatrix} = \begin{bmatrix} -2a & -2b & -2c \\ 3a & 3b & 3c \\ -4a & -4b & -4c \end{bmatrix}$$

$$\mathbf{v}\mathbf{u}^T = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \begin{bmatrix} -2 & 3 & -4 \end{bmatrix} = \begin{bmatrix} -2a & 3a & -4a \\ -2b & 3b & -4b \\ -2c & 3c & -4c \end{bmatrix}$$

Exercises 2.2 (p. 117)

Assignment: Do #9, 10, 1, 3, 5, 7, 13, 15

9. (a) True. (p. 110)

(b) False. The inverse of AB is $B^{-1}A^{-1}$.

(c) False. If $ad - bc \neq 0$, then A is invertible.

(d) True. (p. 111, Theorem 5)

(e) True. (p. 114)

10. (a) False. The product of invertible square matrices is invertible, but the inverse of the product is the product of their inverses *in reverse order*.

(b) True. (p. 113, Theorem 6)

(c) True, because then $ad - bc = 0$.

(d) True. (p. 115, Theorem 7)

(e) False. It's backwards: they reduce I_n to A^{-1}.

1. By the formula,

$$A^{-1} = \frac{1}{(-4)6 - (-5)5} \begin{bmatrix} 6 & -(-5) \\ -(5) & -4 \end{bmatrix} = \frac{1}{1} \begin{bmatrix} 6 & 5 \\ -5 & -4 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ -5 & -4 \end{bmatrix}$$

Alternatively, using the row reduction method is very tedious, but it works, too:

$$\begin{bmatrix} -4 & -5 & 1 & 0 \\ 5 & 6 & 0 & 1 \end{bmatrix} \xrightarrow{R2 \to R2 + \frac{5}{4} R1} \begin{bmatrix} \boxed{-4} & -5 & 1 & 0 \\ 0 & -\frac{1}{4} & \frac{5}{4} & 1 \end{bmatrix}$$

$$\xrightarrow{R2 \to -4R2} \begin{bmatrix} \boxed{-4} & -5 & 1 & 0 \\ 0 & 1 & -5 & -4 \end{bmatrix} \xrightarrow{R1 \to R1 + 5R2} \begin{bmatrix} -4 & 0 & -24 & -20 \\ 0 & \boxed{1} & -5 & -4 \end{bmatrix}$$

$$\xrightarrow{R1 \to -\frac{1}{4} R1} \begin{bmatrix} 1 & 0 & 6 & 5 \\ 0 & \boxed{1} & -5 & -4 \end{bmatrix}$$

3. Using the formula,

$$A^{-1} = \frac{1}{3(13) - (-7)(-6)} \begin{bmatrix} 13 & -(-7) \\ -(-6) & 3 \end{bmatrix} = \frac{1}{-3} \begin{bmatrix} 13 & 7 \\ 6 & 3 \end{bmatrix} = \begin{bmatrix} -13/3 & -7/3 \\ -2 & -1 \end{bmatrix}$$

5. By Theorem 5, the unique solution is

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 6 & 5 \\ -5 & -4 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 40 \\ -33 \end{bmatrix}$$

7. By the formula

$$A^{-1} = \frac{1}{2} \begin{bmatrix} 8 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ -3/2 & 1/2 \end{bmatrix}$$

By Theorem 5, the unique solutions are given by $A^{-1}\mathbf{b}_1$ through $A^{-1}\mathbf{b}_4$ respectively:

$$A^{-1}\mathbf{b}_1 = \begin{bmatrix} 4 & -1 \\ -3/2 & 1/2 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 13 \\ -4 \end{bmatrix}$$

$$A^{-1}\mathbf{b}_2 = \begin{bmatrix} 4 & -1 \\ -3/2 & 1/2 \end{bmatrix} \begin{bmatrix} -2 \\ -2 \end{bmatrix} = \begin{bmatrix} -6 \\ 2 \end{bmatrix}$$

$$A^{-1}\mathbf{b}_3 = \begin{bmatrix} 4 & -1 \\ -3/2 & 1/2 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \end{bmatrix}$$

$$A^{-1}\mathbf{b}_4 = \begin{bmatrix} 4 & -1 \\ -3/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 7 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

13. As with all problems of this type, the idea is to simplify $AB = AC$ by taking advantage of the fact that A is invertible, and so we can "cancel" A (that is, change it into the identiy matrix) by multiplying it on the left or right by A^{-1}. We begin by multiplying both sides of the equation by A^{-1} on the left:

$$\begin{aligned} AB &= AC \\ A^{-1}(AB) &= A^{-1}(AC) \\ (A^{-1}A)B &= (A^{-1}A)C \\ IB &= IC \\ B &= C \end{aligned}$$

If A is *not* invertible, this won't be true in general. In fact, if $A = 0$ is the (noninvertible) zero matrix, then $AB = AC$ for *any* matrices B and C.

15. We actually looked at this in class (but with $ABCDE$ instead of just ABC). Again, the key here is to construct a matrix that will cancel properly. If $(ABC)D = I$, we want the leftmost component of D to cancel C, then the remainder to cancel B, and then the remainder to cancel A. If we choose $D = C^{-1}B^{-1}A^{-1}$, this will work:

$$(ABC)D = (ABC)(C^{-1}B^{-1}A^{-1}) = AB(CC^{-1})B^{-1}A^{-1} = A(BB^{-1})A^{-1} = AA^{-1} = I$$

and similarly

$$D(ABC) = (C^{-1}B^{-1}A^{-1})(ABC) = C^{-1}B^{-1}(A^{-1}A)BC = C^{-1}(B^{-1}B)C = C^{-1}C = I$$

Exercises 2.3 (p. 123)

Assignment: Do #13, 14

13. Note that, for this problem, all matrices are $n \times n$.

(a) True. (Theorem 8 (d) $\Leftrightarrow$ (b))

(b) True. (Theorem 8 (h) $\Leftrightarrow$ (e))

(c) False, if A is noninvertible. (Theorem 8 (a) $\Leftrightarrow$ (g))

(d) True. By Theorem 8 (d) $\Leftrightarrow$ (c), if $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution, A does *not* have n pivot positions. Since it can't have more than n pivot positions, it must have fewer.

(e) True. (Theorem 8 (a) $\Leftrightarrow$ (l))

14. (a) True. (Theorem 8 (j) $\Leftrightarrow$ (k))

(b) True. (Theorem 8 (h) $\Leftrightarrow$ (e))

(c) True. If the equation is consistent for all $\mathbf{b}$, by Theorem 8, A has exactly n pivot positions, one in each column. Therefore, for every augmented matrix $\begin{bmatrix} A & \mathbf{b} \end{bmatrix}$, there are no free variables, so the solution is always unique.

(d) False. It's a trick question. For an $n \times n$ matrix, the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ always maps $\mathbb{R}^n$ *into* $\mathbb{R}^n$. The real question is whether the transformation is *onto* $\mathbb{R}^n$.

(e) True. (Theorem 8 (g) $\Leftrightarrow$ (f))