Homework Set #5 Solutions

Corrections: (Sept. 29) 1.2 #10

Exercises 1.8 (p. 83)

Assignment: Do #23, 24, 1, 3, 6, 12, 15, 17, 19, 21, 25, 27, 29, 31, 32, 35

23. (a) True. (p. 76)

(b) True. (p. 77)

(c) False. *Any* mapping $T\colon \mathbb{R}^n \to \mathbb{R}^m$ maps every vector $\mathbf{x}$ in $\mathbb{R}^n$ onto some vector in $\mathbb{R}^m$. An "onto $\mathbb{R}^m$" mapping is one where every vector $\mathbf{b}$ in $\mathbb{R}^m$ gets mapped onto *by* some vector $\mathbf{x}$ in $\mathbb{R}^n$.

24. (a) False. (p. 77, Theorem 10)

(b) True. (p. 77, Theorem 10)

(c) True. (p. 81)

1. $A = \left[T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) \quad T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) \right] = \begin{bmatrix} 4 & -5 \\ -1 & 3 \\ 2 & -6 \end{bmatrix}$

3. $A = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & T(\mathbf{e}_3) \end{bmatrix} = \begin{bmatrix} 1 & -2 & 3 \\ 4 & 9 & -8 \end{bmatrix}$

6. As T rotates points clockwise through $\pi/2$ radians (or $90°$), note that it rotates the vector $\mathbf{e}_1 = (1,0)$ to $(0,-1)$ and the vector $\mathbf{e}_2 = (0,1)$ to $(1,0)$. (Drawing a picture will help.) From this, we can easily calculate its standard matrix:

$$A = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

We could also use the formula at the top of page 78: since a clockwise angle of $90°$ is equivalent to a counterclockwise angle of $\varphi = 270°$, we can use a calculator to determine that $\sin 270° = -1$ and $\cos 270° = 0$, and the formula gives the same matrix as above.

12. By drawing a picture, we see that $T(\mathbf{e}_1) = (0,-1)$ while $T(\mathbf{e}_2) = (1,0)$. Thus, the standard matrix is

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

15. The trick here is just to see what the first row of the matrix needs to be in order for the first entry of the right-hand side to come out right. Repeat for the second and third rows.

$$\begin{bmatrix} 0 & 2 & -1 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

17. Use the same technique as in problem 15, by writing

$$\begin{bmatrix} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_2 + x_3 \\ x_3 + x_4 \\ 0 \end{bmatrix}$$

to get the matrix

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

19. Using the technique of problem 17, we write

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_2 - x_3 \\ x_1 + 4x_2 + x_3 \end{bmatrix}$$

to get the matrix

$$\begin{bmatrix} 0 & 3 & -1 \\ 1 & 4 & 1 \end{bmatrix}$$

21. If $T(x_1, x_2) = (-2, -5)$, then, by the definition of T, we have the system of equations

$$\begin{cases} x_1 + x_2 = -2 \\ 4x_1 + 7x_2 = -5 \end{cases}$$

The reduced echelon form of the augmented matrix is

$$\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 1 \end{bmatrix}$$

giving the unique solution $(x_1, x_2) = (-3, 1)$. We can check our answer by checking that $T(-3, 1) = (-2, -5)$ which it does.

25. By Theorem 12, T is one-to-one iff the columns of A are linearly independent where

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 4 & 9 & -8 \end{bmatrix}$$

Since these are 3 vectors in $\mathbb{R}^2$, by Theorem 8, they must be linearly dependent. Thus, T is not one-to-one.

27. By Theorem 12, T is onto $\mathbb{R}^2$ if the columns of A span $\mathbb{R}^2$. By Theorem 4, this is true iff A has a pivot position in every row:

$$\begin{bmatrix} 1 & -2 & 3 \\ 4 & 9 & -8 \end{bmatrix} \xrightarrow{R2 \to R2 - 4R1} \begin{bmatrix} 1 & -2 & 3 \\ 0 & 17 & -20 \end{bmatrix}$$
$$\Uparrow$$

Since A has a pivot position in every row, T *is* onto $\mathbb{R}^2$.

29. The matrix of this transformation, already in echelon form, is

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

It's columns are linearly dependent (for, if we formed the augmented matrix $[A \quad \mathbf{0}]$, the fourth column would not be a pivot column, giving a free variable and a nontrivial solution to the homogeneous equation $A\mathbf{x} = \mathbf{0}$), so by Theorem 12, T is not one-to-one. Since it doesn't have a pivot position in every row, either, by Theorem 4 and Theorem 12, T is not onto $\mathbb{R}^4$.

31. *"T is one-to-one if and only if A has n pivot columns."* By Theorem 12, T is one-to-one iff the columns of A are linearly independent. This is true iff $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. This is true if the augmented matrix $[A \quad \mathbf{0}]$ has no free variables—in other words, if it has a pivot column in every column but the rightmost column. And that's true iff A has all n of its columns as pivot columns.

32. *"T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if A has m pivot columns."* By Theorem 12, T is onto iff the columns of A span all of $\mathbb{R}^m$. By Theorem 6, this is true iff all m of A's rows have pivot positions. And that's true iff A has exactly m pivot columns.

35. By problem 32, T is onto $\mathbb{R}^m$ iff A has exactly m pivot columns. This can happen only of the number of columns of A is at least m, so we need $n \geq m$. By problem 31, T is one-to-one iff A has exactly n pivot columns and, so, exactly n pivot positions. But, this requires at least n rows in the matrix, so $m \geq n$.

Exercises 2.1 (p. 107)

Assignment: Do #15, 16, 1, 2, 3, 7, 8, 9, 10, 11, 12

15. (a) False. The correct definition is $AB = [A\mathbf{b}_1 \quad A\mathbf{b}_2]$. The quantities $\mathbf{a}_1\mathbf{b}_1$ and $\mathbf{a}_2\mathbf{b}_2$ aren't even defined!

(b) False. It's backwards: each column of AB is a linear combination of the columns of A using the weights from the corresponding column of B.

(c) True. (p. 104)

(d) True. (p. 106)

(e) False. The transpose of the product of matrices equals the product of their transposes *taken in reverse order.*

16. (a) False. As above, $AB = [A\mathbf{b}_1 \quad A\mathbf{b}_2 \quad A\mathbf{b}_3]$.

(b) True. (p. 104)

(c) False, in general.

(d) False. $(AB)^T = B^T A^T$

(e) True. (p. 106)

1. See back of textbook.

2. (a) $A + B = \begin{bmatrix} 6 & 4 & 0 \\ 4 & 2 & 2 \end{bmatrix}$

(b) $3C - E$ is undefined because $3C$ is a 2×2 matrix (the same size as C) but E is a 2×1 matrix, and you can't add two matricies with different sizes.

(c) $CB = \begin{bmatrix} 19 & -8 & 1 \\ 4 & -16 & -4 \end{bmatrix}$

(d) EB is undefined since a 2×1 and a 2×3 matrix can't be multiplied: the number of columns of the first isn't the same as the number of rows of the second.

3.

$$3I_2 - A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 4 & -1 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -3 & 5 \end{bmatrix}$$

$$(3I_2)A = 3(I_2 A) = 3A = \begin{bmatrix} 12 & -3 \\ 9 & -6 \end{bmatrix}$$

7. We need $3 \times \boxed{5 \quad n} \times p$ to match in the middle and give 3×7. So, $n = 5$ and $p = 7$, giving B a 5×7 matrix.

8. The number of rows of BA is equal to the number of rows of B, so 2.

9. Calculating $(CD)E$, it requires 8 multiplications to calculate CD as in problem 1, and it requires another 4 multiplications to calculate the product of CD and E:

$$(CD)E = \begin{bmatrix} -7 & 4 \\ -4 & 0 \end{bmatrix} \begin{bmatrix} 7 \\ -3 \end{bmatrix} = \begin{bmatrix} -61 \\ -28 \end{bmatrix}$$

for a total of 12 multiplications.

Calculating $C(DE)$ instead, it requires 4 multiplications to calculate DE:

$$DE = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ -3 \end{bmatrix} = \begin{bmatrix} 7 \\ -17 \end{bmatrix}$$

and another 4 multiplications to finish it off:

$$C(DE) = \begin{bmatrix} 1 & 4 \\ -4 & 0 \end{bmatrix} \begin{bmatrix} 7 \\ -17 \end{bmatrix} = \begin{bmatrix} -61 \\ -28 \end{bmatrix}$$

for a total of only 8 multiplications.

10. Since

$$AB = \begin{bmatrix} 1 & 12-4k \\ -30 & -20+k \end{bmatrix} \qquad BA = \begin{bmatrix} 1 & -24 \\ 15-5k & -20+k \end{bmatrix}$$

we need to find a k that satisfies $12 - 4k = -24$ and $15 - 5k = -30$. ~~Unfortunately, no k satisfies both equations, so there is no k that makes these matrices commute (satisfy $AB = BA$).~~ *Correction:* $k = 9$ satisfies both equations, so this value will make the matrices commute.

11. See back of textbook.

12. Let $B = [b_{ij}]$. Then, we want

$$\begin{bmatrix} 2 & -6 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} 2b_{11} - 6b_{21} & 2b_{12} - 6b_{22} \\ -b_{11} + 3b_{21} & -b_{12} + 3b_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

For the first column, picking $b_{11} = 3$ and $b_{21} = 1$ seems to work (that is, it makes the first column of the middle matrix all zeros). For the second column, we can pick $b_{12} = 6$ and $b_{22} = 2$, and that zeroes out the second column of the middle matrix. The final matrix is

$$B = \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix}$$