Homework Set #4 Solutions

Exercises 1.4 (p. 46)

Assignment: Do #13, 15, 17, 19, 22, 25

13.

$$2\begin{bmatrix} 4 \\ -7 \end{bmatrix} + 6\begin{bmatrix} -2 \\ 1 \end{bmatrix} - \begin{bmatrix} -5 \\ -8 \end{bmatrix} + 0\begin{bmatrix} 9 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

15.

$$\begin{bmatrix} -4 & 3 \\ 1 & -5 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}$$

17. Equivalently, does $A\mathbf{x} = \mathbf{u}$ have a solution? The augmented matrix can be reduced as follows:

$$\begin{array}{c}\begin{bmatrix} 3 & 5 & -5 \\ 1 & 1 & -3 \\ -2 & -8 & -6 \end{bmatrix} \\ \Uparrow \end{array} \xrightarrow{R2 \to R2 - \frac{1}{3}R1} \begin{array}{c}\begin{bmatrix} 3 & 5 & -5 \\ 0 & -\frac{2}{3} & -\frac{4}{3} \\ -2 & -8 & -6 \end{bmatrix} \\ \Uparrow \end{array}$$

$$\xrightarrow{R3 \to R3 + \frac{2}{3}R1} \begin{array}{c}\begin{bmatrix} 3 & 5 & -5 \\ 0 & -\frac{2}{3} & -\frac{4}{3} \\ 0 & -\frac{14}{3} & -\frac{28}{3} \end{bmatrix} \\ \Uparrow \end{array} \xrightarrow{R3 \to R3 - 7R2} \begin{array}{c}\begin{bmatrix} 3 & 5 & -5 \\ 0 & -\frac{2}{3} & -\frac{4}{3} \\ 0 & 0 & 0 \end{bmatrix} \\ \Uparrow \end{array}$$

The equation is consistent, so $\mathbf{u}$ *is* in the plane spanned by the columns.

19. Reducing the augmented matrix:

$$\begin{array}{c}\begin{bmatrix} -3 & 1 & b_1 \\ 6 & -2 & b_2 \end{bmatrix} \\ \Uparrow \end{array} \xrightarrow{R2 \to R2 + 2R1} \begin{array}{c}\begin{bmatrix} -3 & 1 & b_1 \\ 0 & 0 & b_2 + 2b_1 \end{bmatrix} \\ \Uparrow \end{array}$$

This is consistent iff $b_2 + 2b_1 = 0$. That is, it's consistent for all $\mathbf{b}$ satisfying $b_2 = -2b_1$. We may write this set in parametric vector form:

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ -2b_1 \end{bmatrix} = b_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad b_1 \text{ free}$$

and we see that it is consistent for all scalar multiples of $(1, -2)$.

22. If A has four pivot columns, it has four pivot positions. However, each pivot position must occur in a different row. (Since they are the positions of leading entries of an echelon form, you can't have two in the same row.) Therfore, A has four rows containing a pivot position.

25. A single row operation suffices to put the matrix in echelon form:

$$\begin{bmatrix} 0 & 0 & 2 \\ 0 & -5 & 1 \\ 4 & 6 & -3 \end{bmatrix} \xrightarrow{R1 \leftrightarrow R2} \begin{bmatrix} \boxed{4} & 6 & -3 \\ 0 & \boxed{-5} & 1 \\ 0 & 0 & \boxed{2} \end{bmatrix}$$

The pivot positions are as indicated. Since A has a pivot position in every row, by Theorem 4, its columns span $\mathbb{R}^3$.

Exercises 1.5 (p. 55)

Assignment: Do #1, 3, 5, 15, 17, 26–33

1. The augmented matrix is

$$\begin{bmatrix} 1 & -5 & 9 & 0 \\ -1 & 4 & -3 & 0 \\ 2 & -8 & 9 & 0 \end{bmatrix}$$

We can reduce this matrix to an echelon form with two row operations:

$$\xrightarrow{R3 \to R3+2R2} \begin{bmatrix} 1 & -5 & 9 & 0 \\ -1 & 4 & -3 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix} \xrightarrow{R2 \to R2+R1} \begin{bmatrix} 1 & -5 & 9 & 0 \\ 0 & -1 & 6 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

(Using the "usual" algorithm, it takes three row operations.)

From this echelon form, the general solution has no free variables, so the original system has only the trivial solution: there are no nontrivial solutions.

3. This only takes one row operation:

$$\begin{bmatrix} 5 & -1 & 3 & 0 \\ 4 & -3 & 7 & 0 \end{bmatrix} \xrightarrow{R2 \to R2 - \frac{4}{5}R1} \begin{bmatrix} 5 & -1 & 3 & 0 \\ 0 & -11/5 & 23/5 & 0 \end{bmatrix}$$

The echelon form is ugly, but its general solution has a free variable, so the system has a nontrivial solution.

5. Reducing the augmented matrix, we have

$$\begin{bmatrix} 1 & -3 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ -2 & 3 & 7 & 0 \end{bmatrix} \xrightarrow{R3 \to R3 + 2R1} \begin{bmatrix} 1 & -3 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -3 & 3 & 0 \end{bmatrix}$$
$$\Uparrow \qquad\qquad\qquad\qquad \Uparrow$$

$$\xrightarrow{R3 \to R3 + 3R2} \begin{bmatrix} \boxed{1} & -3 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R1 \to R1 + 3R2} \begin{bmatrix} \boxed{1} & 0 & -5 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\Uparrow \qquad\qquad\qquad\qquad \Uparrow$$

This gives a general solution

$$\begin{cases} x_1 = 5x_3 \\ x_2 = x_3 \\ x_3 \text{ free} \end{cases}$$

which we can rewrite in parametric vector form like so:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}, \quad x_3 \text{ free}$$

15. This looks a little strange, but we can solve it the way we solve all linear systems. It's a single equation in three unknowns, so its augmented matrix is:

$$\begin{bmatrix} 1 & -4 & 3 & 0 \end{bmatrix}$$

It's already in echelon form with a single basic variable, so we can write its general solution as:

$$\begin{cases} x_1 = 4x_2 - 3x_3 \\ x_2,\ x_3 \text{ free} \end{cases}$$

(Note that this answer is no different than if we'd rearranged the original, single equation to express x_1 as a linear combination of x_2 and x_3 and then stated that x_2 and x_3 could be treated as free variables.) We can also express this solution set in vector parametric form:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4x_2 - 3x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -3x_3 \\ 0 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \qquad x_2,\ x_3 \text{ free}$$

As for the nonhomogeneous equation, if we observe that $(7, 0, 0)$ is a solution to $x_1 - 4x_2 + 3x_3 = 7$, we can apply Theorem 6 to conclude that its solution set is

$$\mathbf{x} = \begin{bmatrix} 7 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \qquad x_2,\ x_3 \text{ free}$$

Any other solution to the nonhomogeneous system would have worked in place of $(7, 0, 0)$.

Comparing the two solution sets, they are parallel planes. The former passes through the origin. The latter passes through $(7, 0, 0)$.

17. The parametric equation of the line through **a** parallel to **b** always looks like:

$$\mathbf{x} = \mathbf{a} + c\mathbf{b}, \quad c \in \mathbb{R}$$

(To remember which vector has the constant c, take the constant to be zero, and note that this gives $\mathbf{x} = \mathbf{a}$. That is, the line passes through the point **a**, not **b**, so we must have written it down correctly.)

For this example, we just plug in the values for the vectors:

$$\mathbf{x} = \begin{bmatrix} 3 \\ -8 \end{bmatrix} + c \begin{bmatrix} -1 \\ 5 \end{bmatrix}, \quad c \in \mathbb{R}$$

26. This is a consequence of Theorem 6. If $A\mathbf{x} = \mathbf{b}$ has a solution (say $\mathbf{p}$), then Theorem 6 applies, and the solution set of $A\mathbf{x} = \mathbf{b}$ consists of all solutions $\mathbf{x} = \mathbf{p} + \mathbf{v}_h$ where $\mathbf{v}_h$ can vary over all solutions of $A\mathbf{x} = \mathbf{0}$.

If the solution set of $A\mathbf{x} = \mathbf{0}$ consists of only the trivial solution $\mathbf{0}$, then the solution set of $A\mathbf{x} = \mathbf{b}$ consists of only $\mathbf{x} = \mathbf{p} + \mathbf{0} = \mathbf{p}$. Conversely, if $A\mathbf{x} = \mathbf{0}$ also has a nontrivial solution (say $\mathbf{q}$), then $A\mathbf{x} = \mathbf{b}$ has another solution $\mathbf{x} = \mathbf{p} + \mathbf{q}$ distinct from $\mathbf{x} = \mathbf{p}$.

27. This is a great problem. Going back to our basic concepts, the augmented matrix is

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

It's already in echelon form (if you read the definition carefully). It also has no pivot positions. Therefore, it is consistent and there are no basic variables: that is, all the variables are free! The general solution is:

$$\left\{ x_1,\ x_2,\ x_3 \text{ free} \right.$$

In other words, *every* vector in $\mathbb{R}^3$ is a solution.

You don't really have to go through all that work, though. You could observe that, for the matrix A of all zeros, $A\mathbf{v} = 0$ holds for all $\mathbf{v} \in \mathbb{R}^3$, and that would show that every vector in $\mathbb{R}^3$ is a solution.

28. No. Suppose that the solution set *was* a plane through the origin. That would mean that the origin was one of the solutions. Therefore, we would have $A\mathbf{0} = \mathbf{b}$. However, $A\mathbf{0}$ always equals $\mathbf{0}$ for any matrix. Therefore, if $\mathbf{b} \neq \mathbf{0}$, this can't happen.

29. A 3×3 matrix with three pivot positions has a pivot position in every row and every column.

(a) The augmented matrix $[A \quad \mathbf{0}]$ will have a pivot position in the first three columns (and no pivot position in the rightmost column—homogeneous systems can never give rise to an augmented matrix with a pivot position in the rightmost column because they are always consistent). Therefore, it has no free variables, and $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(b) Since there's a pivot position in every row, $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b} \in \mathbb{R}^3$ (by Theorem 4).

30. (a) The augmented matrix $[A \quad \mathbf{0}]$ will have two pivot positions among its left three columns. Therefore, it has one free variable, and $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution.

(b) Since there's a pivot position in only two out of three rows, $A\mathbf{x} = \mathbf{b}$ does not have a solution for every $\mathbf{b} \in \mathbb{R}^3$ (by Theorem 4).

31. (a) The three-columned augmented matrix $[A \quad \mathbf{0}]$ will have two pivot positions among its left two columns. Therefore, it has no free variables, and $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(b) As in 30(b), since there's a pivot position in only two out of three rows, $A\mathbf{x} = \mathbf{b}$ does not have a solution for every $\mathbf{b} \in \mathbb{R}^3$ (by Theorem 4).

32. (a) The five-columned augmented matrix $[A \quad \mathbf{0}]$ will have pivot positions in two of its left four columns. Therefore, it has two free variables, and $A\mathbf{x} = \mathbf{0}$ will have a nontrivial solution.

(b) Since there's a pivot position in every row, by Theorem 4, $A\mathbf{x} = \mathbf{b}$ will have a solution for every $\mathbf{b} \in \mathbb{R}^2$.

33. Note that the second column is twice the first column. If you calculate the difference $2\mathbf{a}_1 - \mathbf{a}_2$, you'll see that this evaluates to $\mathbf{0}$. This gives the nontrivial solution $(2, -1)$.

Exercises 1.6 (p. 64)

Assignment: Do #27, 28, 1, 3, 7, 15, 19, 21, 23, 26, 31, 33, 35, 37, 39

27. (a) False. The columns of matrix A are linearly independent if the equation $A\mathbf{x} = \mathbf{0}$ has *only* the trivial solution.

(b) False. By Theorem 7, at least one vector will be a linear combination of the others, but it's not necessary that *every* vector be a linear combination of the others.

(c) True. Theorem 8 (or Figure 3) applies here.

(d) True. (p. 62)

28. (a) True. (p. 61)

(b) False. See the warning at the top of page 63. The set $\{(0, 0)\}$ contains only one vector with two entries, but it's linearly dependent.

(c) True. Since $\mathbf{z}$ is in the span of the other two, it can be expressed as a linear combination of them, and Theorem 7 applies.

(d) False, for the same reason (b) is false. The set $\{(0, 0)\}$ is linearly dependent, but it contains fewer vectors than there are entries in each vector.

1. Let's determine if the linear system with augmented matrix:

$$\begin{bmatrix} 3 & -3 & 6 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 3 & 0 & 0 \end{bmatrix}$$

has a nontrivial solution. Reducing the augmented matrix:

$$\begin{bmatrix} 3 & -3 & 6 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 3 & 0 & 0 \end{bmatrix} \xrightarrow{R3 \to R3 - \frac{3}{2}R2} \begin{bmatrix} 3 & -3 & 6 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & -6 & 0 \end{bmatrix}$$

There are no free variables, so there is no nontrivial solution. Therefore, the given set is linearly independent.

3. Note that the second vector is twice the first vector. By Theorem 7, these vectors are linearly dependent.

7. There are four columns with only three entries each. By Theorem 8, they must be linearly dependent.

15. Reducing the appropriate augmented matrix

$$\begin{bmatrix} 1 & -2 & -1 & 0 \\ 3 & -4 & 1 & 0 \\ -3 & 1 & h & 0 \end{bmatrix} \xrightarrow{R2 \to R2 - 3R1} \begin{bmatrix} 1 & -2 & -1 & 0 \\ 0 & 2 & 4 & 0 \\ -3 & 1 & h & 0 \end{bmatrix}$$

$$\xrightarrow{R3 \to R3 + 3R1} \begin{bmatrix} 1 & -2 & -1 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & -5 & h-3 & 0 \end{bmatrix} \xrightarrow{R3 \to R3 + \frac{5}{2}R2} \begin{bmatrix} 1 & -2 & -1 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & h+7 & 0 \end{bmatrix}$$

The vectors are linearly dependent iff $h + 7 = 0$ (so that x_3 is a free variable). Therefore, the vectors are linearly dependent iff $h = -7$.

19. There are four vectors with only two entries each. By Theorem 8, they must be linearly dependent.

21. Because this set contains the zero vector, it is linearly dependent by Theorem 9.

23. Two vectors are linearly dependent iff one is a scalar multiple of the other. Neither of these vectors is a scalar multiple of the other, so they must be linearly independent.

26. For A, we need a matrix with pivot positions in both columns. Then, the augmented matrix will have no free variables. Therefore, if we take

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

then $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

For B, we need a matrix that doesn't have pivot positions in both columns. Then, the augmented matrix will have at least one free variable, and there will be a nontrivial solution. An easy choice is

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

For this matrix, *every* vector x is a solution to $B\mathbf{x} = \mathbf{0}$.

31. True. Since one vector ($\mathbf{v}_3$) can be written as a linear combination of the others, the whole set is linearly dependent by Theorem 7.

33. False! We have to be careful here. A set of two vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent if *neither* vector is a scalar multiple of the other. We're told that $\mathbf{v}_1$ isn't a scalar multiple of $\mathbf{v}_2$, but what if $\mathbf{v}_1 = (1, 1)$ and $\mathbf{v}_2 = (0, 0)$? In this case, the set is linearly dependent by Theorem 9, and yet $\mathbf{v}_1$ is not a scalar multiple of $\mathbf{v}_2$. What's happened is that is that even though $\mathbf{v}_1$ is not a scalar multiple of $\mathbf{v}_2$, the vector $\mathbf{v}_2$ *is* a scalar multiple (the zero multiple) of $\mathbf{v}_1$.

35. True. If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent, there's a linear dependence relation among those three vectors (that is, a linear combination with at least one nonzero weight that equals $\mathbf{0}$). We can use this same linear dependence relation as a linear dependence relation among the four vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$, showing that they are linearly dependent, too.

37. The columns of A are linearly independent if and only if it has a pivot position in every column (so that the augmented matrix $[A \quad \mathbf{0}]$ will have no free variables). Thus, we need exactly 5 pivot positions.

39. If the property is true for *all* $\mathbf{b} \in \mathbb{R}^m$, then it's true for $\mathbf{b} = \mathbf{0}$. Therefore, the equation $A\mathbf{x} = \mathbf{0}$ has at most one solution, so we know it must have only the trivial solution. That is, $x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n = \mathbf{0}$ has only the trivial solution, so by definition the columns of A are linearly independent.

Exercises 1.7 (p. 73)

Assignment: Do #23, 24, 1, 3, 7, 9, 11, 13, 15, 19, 21, 25, 29

23. (a) True. A linear transformation is a function that satisfies the special properties of the definition on p. 70.

(b) False. The domain of T is $\mathbb{R}^5$ and the codomain is $\mathbb{R}^3$.

(c) True. (p. 71)

(d) I don't know what the author's trying to ask here. The superposition principle is a physical description of a *property* of a linear transformations but not of a linear transformation per se.

24. (a) True. (p. 70)

(b) False. That would be an existence question. A uniqueness question would be "is there more than one vector whose image under T is **c**."

(c) True. (p. 70)

1.

$$T(\mathbf{u}) = T\left(\begin{bmatrix}1\\5\end{bmatrix}\right) = \begin{bmatrix}3&0\\0&3\end{bmatrix}\begin{bmatrix}1\\5\end{bmatrix} = \begin{bmatrix}3\\15\end{bmatrix}$$

$$T(\mathbf{v}) = T\left(\begin{bmatrix}-4\\-1\end{bmatrix}\right) = \begin{bmatrix}3&0\\0&3\end{bmatrix}\begin{bmatrix}-4\\-1\end{bmatrix} = \begin{bmatrix}-12\\-3\end{bmatrix}$$

3. We need to determine if $A\mathbf{x} = \mathbf{b}$ has a solution and if this solution is unique. Reducing the augmented matrix, we have:

$$\begin{bmatrix}1&0&-1&0\\3&1&-5&-5\\-4&2&1&-6\end{bmatrix} \xrightarrow{R2\to R2-3R1} \begin{bmatrix}1&0&-1&0\\0&1&-2&-5\\-4&2&1&-6\end{bmatrix} \xrightarrow{R3\to R3+4R1} \begin{bmatrix}1&0&-1&0\\0&1&-2&-5\\0&2&-3&-6\end{bmatrix}$$

$$\xrightarrow{R3\to R3-2R2} \begin{bmatrix}\boxed{1}&0&-1&0\\0&\boxed{1}&-2&-5\\0&0&1&4\end{bmatrix} \xrightarrow{R1\to R1+R3} \begin{bmatrix}\boxed{1}&0&0&4\\0&\boxed{1}&-2&-5\\0&0&1&4\end{bmatrix} \xrightarrow{R2\to R2+2R3} \begin{bmatrix}\boxed{1}&0&0&4\\0&\boxed{1}&0&3\\0&0&1&4\end{bmatrix}$$

This has the unique solution $(4, 3, 4)$. Therefore, this is the unique $\mathbf{x}$ whose image under T is the given $\mathbf{b}$.

7. $a = 5$ and $b = 7$

9. We want to find all $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{0}$; that is, we want to find the solution set of this equation. Reducing the augmented matrix, we have:

$$\begin{bmatrix} 1 & 3 & 4 & -3 & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 3 & 7 & 6 & -5 & 0 \end{bmatrix} \xrightarrow{R3 \to R3 - 3R1} \begin{bmatrix} 1 & 3 & 4 & -3 & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & -2 & -6 & 4 & 0 \end{bmatrix}$$

$$\xrightarrow{R3 \to R3 + 2R2} \begin{bmatrix} \boxed{1} & 3 & 4 & -3 & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R1 \to R1 - 3R2} \begin{bmatrix} \boxed{1} & 0 & -5 & 3 & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This gives the general solution

$$\begin{cases} x_1 = 5x_3 - 3x_4 \\ x_2 = -3x_3 + 2x_4 \\ x_3,\ x_4 \text{ free} \end{cases}$$

which gives the set of all $\mathbf{x}$ mapped into the zero vector by $\mathbf{x} \mapsto A\mathbf{x}$.

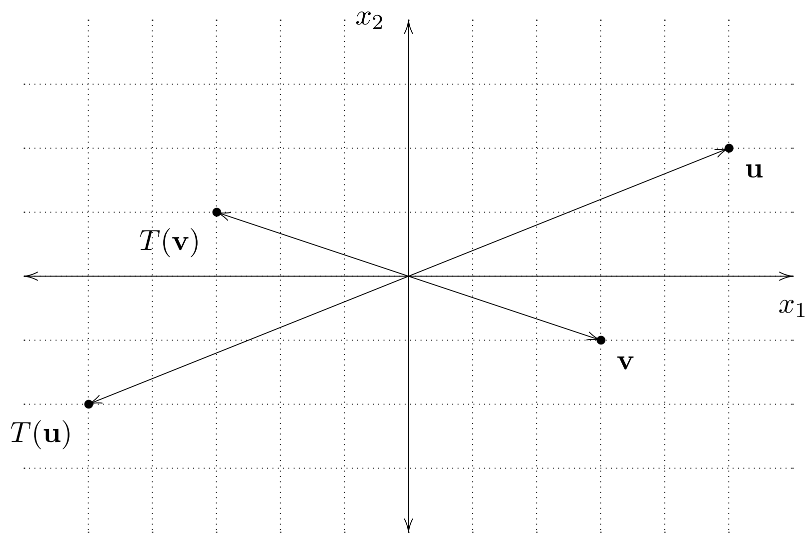
11. In other words, is there a solution to $A\mathbf{x} = \mathbf{b}$? Reducing the augmented matrix, we have:

$$\begin{bmatrix} 1 & 3 & 4 & -3 & 1 \\ 0 & 1 & 3 & -2 & -1 \\ 3 & 7 & 6 & -5 & 7 \end{bmatrix} \xrightarrow{R3 \to R3 - 3R1} \begin{bmatrix} 1 & 3 & 4 & -3 & 1 \\ 0 & 1 & 3 & -2 & -1 \\ 0 & -2 & -6 & 4 & 4 \end{bmatrix}$$

$$\xrightarrow{R3 \to R3 + 2R2} \begin{bmatrix} 1 & 3 & 4 & -3 & 1 \\ 0 & 1 & 3 & -2 & -1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

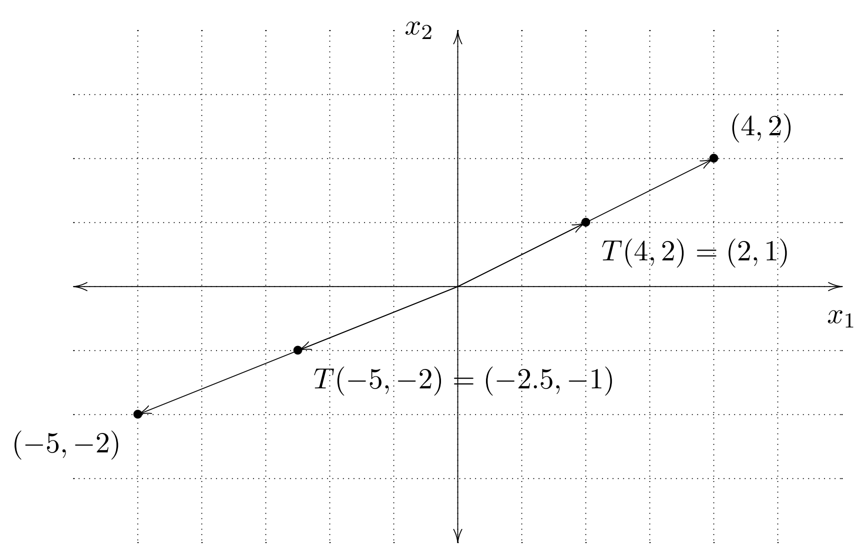
Therefore, the system is inconsistent, and the given $\mathbf{b}$ is *not* in the range of the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$

13. (a) Note that $T(\mathbf{u}) = (-5, -2)$ and $T(\mathbf{v}) = (-3, 1)$.



(b) It rotates each vector 180°. Equivalently, it reflects each vector through the origin (so it comes out the same length on the other side).

15. Note that $T((4, 2)) = (2, 1)$ and $T((-5, -2)) = (-2.5, -1)$.



It contracts each vector halfway towards the origin.

19.

$$T(2\mathbf{u}) = 2T(\mathbf{u}) = 2(2,0) = (4,0)$$
$$T(3\mathbf{v}) = 3T(\mathbf{v}) = 3(1,-4) = (3,-12)$$
$$T(2\mathbf{u}+3\mathbf{v}) = T(2\mathbf{u}) + T(3\mathbf{v}) = (4,0) + (3,-12) = (7,-12)$$

21. We can write

$$\begin{bmatrix} 7 \\ 6 \end{bmatrix} = 7\mathbf{e}_1 + 6\mathbf{e}_2$$

Therefore, we have

$$T\left(\begin{bmatrix} 7 \\ 6 \end{bmatrix}\right) = 7T(\mathbf{e}_1) + 6T(\mathbf{e}_2) = 7\begin{bmatrix} 3 \\ -5 \end{bmatrix} + 6\begin{bmatrix} -2 \\ 7 \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \end{bmatrix}$$

More generally,

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = x_1T(\mathbf{e}_1) + x_2T(\mathbf{e}_2) = x_1\begin{bmatrix} 3 \\ -5 \end{bmatrix} + x_2\begin{bmatrix} -2 \\ 7 \end{bmatrix} = \begin{bmatrix} 3x_1 - 2x_2 \\ -5x_1 + 7x_2 \end{bmatrix}$$

25. Observe that each point of the line has image

$$T(\mathbf{x}) = T(\mathbf{p} + t\mathbf{v}) = T(\mathbf{p}) + tT(\mathbf{v})$$

The image of all these points forms the line through $T(\mathbf{p})$ in the direction of $T(\mathbf{v})$, as long as $T(\mathbf{v}) \neq 0$. If $T(\mathbf{v}) = 0$, then we have $T(\mathbf{x}) = T(\mathbf{p})$ for all points $\mathbf{x}$ on the line—that is, the image of the whole line is just the point $T(\mathbf{p})$.

29. (a) For $f(x) = mx$, note that

$$f(x+y) = m(x+y) = mx + my = f(x) + f(y)$$

and

$$f(cx) = m(cx) = c(mx) = cf(x)$$

so f is linear.

(b) For $f(x) = mx + b$ with $b \neq 0$, note that $f(0) = b \neq 0$. But we showed in class that all linear transformations T have $T(0) = 0$. (We actually showed it for linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$, but the same argument applies for $T\colon \mathbb{R} \to \mathbb{R}$.) Thus, this f cannot be a linear transformation.

(c) Good question. We call f a linear *function* for the same reason we named linear equations the way we did: they involve sums of constants and variables "naked" except possibly for multiplication by a constant. However, we reserve the term "linear transformation" for transformations that satisfy the definition on p. 70.

This is a bit of a terminological mess. Even though "transformation" and "function" mean the same thing, "linear transformation" and "linear function" mean different things! Let's pretend I never assigned this part of the question.