Homework Set #3 Solutions

Exercises 1.1 (p. 10)

Assignment: Do #20, 26

20. The augmented matrix can be reduced as follows:

$$\begin{bmatrix} 1 & 0 & -2 & 0 & -1 \\ 0 & 1 & 0 & -1 & 2 \\ 0 & -3 & 2 & 0 & 0 \\ -4 & 0 & 0 & 7 & -5 \end{bmatrix} \xrightarrow{R4 \to R4+4R1} \begin{bmatrix} 1 & 0 & -2 & 0 & -1 \\ 0 & 1 & 0 & -1 & 2 \\ 0 & -3 & 2 & 0 & 0 \\ 0 & 0 & -8 & 7 & -9 \end{bmatrix}$$

$$\xrightarrow{R3 \to R3+3R2} \begin{bmatrix} 1 & 0 & -2 & 0 & -1 \\ 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 2 & -3 & 6 \\ 0 & 0 & -8 & 7 & -9 \end{bmatrix} \xrightarrow{R4 \to R4+4R3} \begin{bmatrix} 1 & 0 & -2 & 0 & -1 \\ 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 2 & -3 & 6 \\ 0 & 0 & 0 & -5 & 15 \end{bmatrix}$$

The rightmost column isn't a pivot column, so the system is consistent.

26.

$$\begin{bmatrix} 2 & 5 & -3 & g \\ 4 & 7 & -4 & h \\ -6 & -3 & 1 & k \end{bmatrix} \xrightarrow{R2 \to R2-2R1} \begin{bmatrix} 2 & 5 & -3 & g \\ 0 & -3 & 2 & h-2g \\ -6 & -3 & 1 & k \end{bmatrix} \xrightarrow{R3 \to R3+3R1} \begin{bmatrix} 2 & 5 & -3 & g \\ 0 & -3 & 2 & h-2g \\ 0 & 12 & -8 & 3g+k \end{bmatrix}$$

$$\begin{bmatrix} 2 & 5 & -3 & g \\ 0 & -3 & 2 & h-2g \\ 0 & 12 & -8 & 3g+k \end{bmatrix} \xrightarrow{R3 \to R3+4R2} \begin{bmatrix} 2 & 5 & -3 & g \\ 0 & -3 & 2 & h-2g \\ 0 & 0 & 0 & -5g+4h+k \end{bmatrix}$$

This system is consistent if and only if the bottom-right entry is zero, that is if and only if $-5g + 4h + k = 0$.

Exercises 1.2 (p. 25)

Assignment: Do #12, 22, 32

12.

$$\begin{bmatrix} 1 & 2 & 4 \\ -2 & -3 & -5 \\ 2 & 1 & -1 \end{bmatrix} \xrightarrow{R2 \to R2+2R1} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 2 & 1 & -1 \end{bmatrix} \xrightarrow{R3 \to R3-2R1} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & -3 & -9 \end{bmatrix}$$
$$\Uparrow \qquad\qquad \Uparrow \qquad\qquad \Uparrow$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & -3 & -9 \end{bmatrix} \xrightarrow{R3 \to R3+3R2} \begin{bmatrix} \boxed{1} & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R1 \to R1-2R2} \begin{bmatrix} \boxed{1} & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$
$$\Uparrow \qquad\qquad \Uparrow \qquad\qquad \Uparrow$$

The general solution is

$$\begin{cases} x_1 = -2 \\ x_2 = 3 \end{cases}$$

22.

$$\begin{bmatrix} 1 & -3 & 1 \\ 2 & h & k \end{bmatrix} \xrightarrow{R2 \to R2-2R1} \begin{bmatrix} 1 & -3 & 1 \\ 0 & h+6 & k-2 \end{bmatrix}$$
$$\Uparrow \qquad\qquad \Uparrow$$

(a) To get a system with no solution, we must have $h + 6 = 0$ and $k - 2 \neq 0$. That is, we need $h = -6$ but $k \neq 2$.

(b) To get a unique solution, we need $h + 6 \neq 0$ and $k - 2 \neq 0$. That is, we need $h \neq -6$ and $k \neq 2$.

(c) Finally, to get an infinite number of solutions, we need the bottom row to be completely zero. That is, we need $h = -6$ and $k = 2$.

32. An underdetermined system has fewer equations than unknowns. In order for the system to be inconsistent, we need the equations to be contradictory. If we have two contradictory equations in three unknowns, that should work:

$$\begin{cases} x_1 + x_2 + x_3 = 4 \\ x_1 + x_2 + x_3 = 8 \end{cases}$$

Exercises 1.3 (p. 36)

Assignment: Do #13, 16, 20, 21, 25, 27

13. We need to determine if the vector equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}$ has a solution. Reducing the augmented matrix:

$$\begin{bmatrix} 1 & 0 & 2 & -5 \\ -2 & 5 & 0 & 11 \\ 2 & 5 & 8 & -7 \end{bmatrix} \xrightarrow{R2 \to R2 + 2R1} \begin{bmatrix} 1 & 0 & 2 & -5 \\ 0 & 5 & 4 & 1 \\ 2 & 5 & 8 & -7 \end{bmatrix} \xrightarrow{R3 \to R3 - 2R1} \begin{bmatrix} 1 & 0 & 2 & -5 \\ 0 & 5 & 4 & 1 \\ 0 & 5 & 4 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 & -5 \\ 0 & 5 & 4 & 1 \\ 0 & 5 & 4 & 3 \end{bmatrix} \xrightarrow{R3 \to R3 - R2} \begin{bmatrix} 1 & 0 & 2 & -5 \\ 0 & 5 & 4 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

we see that this equation is inconsistent. That is, there's no linear combination of the columns of A that equals b.

16. Many answers are possible. Here are a few:

- weights 0 and 0 give $0\mathbf{v}_1 + 0\mathbf{v}_2 = (0, 0, 0)$;
- weights 1 and 0 give $\mathbf{v}_1 + 0\mathbf{v}_2 = (-2, 0, 1)$;
- weights 0 and 1 give $0\mathbf{v}_1 + \mathbf{v}_2 = (1, 0, 2)$;
- weights 1 and 1 give $\mathbf{v}_1 + \mathbf{v}_2 = (-1, 0, 3)$;
- weights -1 and 0 give $-\mathbf{v}_1 + 0\mathbf{v}_2 = (2, 0, -1)$;
- weights 0 and -1 give $0\mathbf{v}_1 - \mathbf{v}_2 = (-1, 0, -2)$;
- weights -1 and -1 give $-\mathbf{v}_1 - \mathbf{v}_2 = (1, 0, -3)$;

In fact, it turns out that any vector $(v_1, 0, v_3)$ is in the span, as long as its middle entry is 0.

20. Recall that the span of two vectors in $\mathbb{R}^3$ is a plane (if neither vector is $\mathbf{0}$ and they aren't scalar multiples of each other). In this case, the span is the plane of all points with $x_2 = 0$. It's the plane that includes the x_1 and x_3 axes.

21. As in problem 13, the question is whether the corresponding vector equation has a solution. Reducing the augmented matrix, we have:

$$\begin{bmatrix} 1 & -5 & 3 \\ 3 & -8 & -5 \\ -1 & 2 & h \end{bmatrix} \xrightarrow{R2 \to R2 - 3R1} \begin{bmatrix} 1 & -5 & 3 \\ 0 & 7 & -14 \\ -1 & 2 & h \end{bmatrix} \xrightarrow{R3 \to R3 + R1} \begin{bmatrix} 1 & -5 & 3 \\ 0 & 7 & -14 \\ 0 & -3 & h+3 \end{bmatrix}$$
$$\Uparrow \qquad\qquad \Uparrow \qquad\qquad \Uparrow$$

$$\begin{bmatrix} 1 & -5 & 3 \\ 0 & 7 & -14 \\ 0 & -3 & h+3 \end{bmatrix} \xrightarrow{R3 \to R3 + \frac{3}{7}R2} \begin{bmatrix} 1 & -5 & 3 \\ 0 & 7 & -14 \\ 0 & 0 & h-3 \end{bmatrix}$$
$$\Uparrow \qquad\qquad \Uparrow$$

(Note that in place of the last row operation, we could have first done $R2 \to \frac{1}{7}R2$ which would have simplified the calculations.)

This is consistent if and only if $h = 3$. Therefore, $\mathbf{b}$ is in the span of $\mathbf{a}_1$ and $\mathbf{a}_2$ if and only if $h = 3$.

25. (a) No, $\mathbf{b}$ isn't in $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$. Don't let the fact that we've been talking about spans of vectors confuse you here. This set consists of exactly three vectors, the vectors $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$. The vector $\mathbf{b}$ would only be in this set if it was *equal* to one of these three vectors, and it obviously isn't.

(b) Okay, *now* we have to do some calculation. The vector $\mathbf{b}$ is in W if and only if the associated vector equation has a solution:

$$\begin{bmatrix} 1 & 0 & -4 & 4 \\ 0 & 3 & -2 & 1 \\ -2 & 6 & 3 & -4 \end{bmatrix} \xrightarrow{R3 \to R3 + 2R1} \begin{bmatrix} 1 & 0 & -4 & 4 \\ 0 & 3 & -2 & 1 \\ 0 & 6 & -5 & 4 \end{bmatrix} \xrightarrow{R3 \to R3 - 2R2} \begin{bmatrix} 1 & 0 & -4 & 4 \\ 0 & 3 & -2 & 1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$
$$\Uparrow \qquad\qquad \Uparrow \qquad\qquad \Uparrow$$

Since this system is consistent, **b** can be written as a linear combination of the columns of A. Therefore it's in their span W. Note that the set W is an infinite set: it contains all the scalar multiples of the first vector $(1, 0, -2)$, for example, and there are an infinite number of those.

(c) Observe that $\mathbf{a}_1$ is a linear combination of the columns of A (if you take weights 1, 0, and 0). Therefore, it's in the set W by definition.

27. (a) The vector $5\mathbf{v}_1$ gives the output of mine #1 over a period of five days.

(b) $x_1 \begin{bmatrix} 20 \\ 550 \end{bmatrix} + x_2 \begin{bmatrix} 30 \\ 500 \end{bmatrix} = \begin{bmatrix} 150 \\ 2825 \end{bmatrix}$

(c) This is kind of ugly, even with a calculator, but it goes something like this:

$$\begin{bmatrix} 20 & 30 & 150 \\ 550 & 500 & 2825 \end{bmatrix} \xrightarrow{R2 \to R2 - \frac{55}{2} R1} \begin{bmatrix} 20 & 30 & 150 \\ 0 & -325 & -1300 \end{bmatrix}$$

$$\begin{bmatrix} \boxed{20} & 30 & 150 \\ 0 & -325 & -1300 \end{bmatrix} \xrightarrow{R2 \to -\frac{1}{325} R2} \begin{bmatrix} \boxed{20} & 30 & 150 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{R1 \to R1 - 30R2} \begin{bmatrix} \boxed{20} & 0 & 30 \\ 0 & 1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 20 & 0 & 30 \\ 0 & \boxed{1} & 4 \end{bmatrix} \xrightarrow{R1 \to \frac{1}{20} R1} \begin{bmatrix} 1 & 0 & \frac{3}{2} \\ 0 & \boxed{1} & 4 \end{bmatrix}$$

So, it takes 3/2 of a day's output from mine #1 together with 4 days' output from mine #2 to get the desired total amount of output.

Exercises 1.4 (p. 46)

Assignment: Do #6, 7, 9, 11

6. Using the definition

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix} = r \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ r \end{bmatrix} + \begin{bmatrix} 0 \\ s \\ 0 \end{bmatrix} + \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ s \\ r \end{bmatrix}$$

Using the clever rule for computing $A\mathbf{x}$, we can write:

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix} = \begin{bmatrix} 0r+0s+1t \\ 0r+1s+0t \\ 1r+0s+0t \end{bmatrix} = \begin{bmatrix} t \\ s \\ r \end{bmatrix}$$

7. $\begin{bmatrix} 3 & -1 & 4 \\ -4 & 1 & -5 \\ 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 6 \end{bmatrix}$

9. $x_1 \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 1 \\ -5 \end{bmatrix} + x_3 \begin{bmatrix} -6 \\ 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix}$

11.

$$\begin{bmatrix} 2 & 4 & -6 & 2 \\ 0 & 1 & 3 & 5 \\ -3 & -5 & 7 & -3 \end{bmatrix} \xrightarrow{R3 \to R3 + \frac{3}{2}R1} \begin{bmatrix} 2 & 4 & -6 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 1 & -2 & 0 \end{bmatrix} \xrightarrow{R3 \to R3 - R2} \begin{bmatrix} \boxed{2} & 4 & -6 & 2 \\ 0 & \boxed{1} & 3 & 5 \\ 0 & 0 & -5 & -5 \end{bmatrix}$$

$$\xrightarrow{R3 \to -\frac{1}{5}R3} \begin{bmatrix} \boxed{2} & 4 & -6 & 2 \\ 0 & \boxed{1} & 3 & 5 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{R1 \to R1 + 6R3} \begin{bmatrix} \boxed{2} & 4 & 0 & 8 \\ 0 & \boxed{1} & 3 & 5 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{R2 \to R2 - 3R3} \begin{bmatrix} \boxed{2} & 4 & 0 & 8 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & \boxed{1} & 1 \end{bmatrix}$$

$$\xrightarrow{R1 \to R1 - 4R2} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 2 \\ 0 & 0 & \boxed{1} & 1 \end{bmatrix} \xrightarrow{R1 \to \frac{1}{2}R1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 2 \\ 0 & 0 & \boxed{1} & 1 \end{bmatrix}$$

The general solution is

$$\begin{cases} x_1 = 0 \\ x_2 = 2 \\ x_3 = 1 \end{cases}$$

which we can rewrite (in parametric vector form, really) as

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

Exercises 1.5 (p. 55)

Assignment: Do #21, 22

21. (a) True. (p. 48) It always has the solution $\mathbf{0}$.

(b) False. An explicit description of the solution set of $A\mathbf{x} = \mathbf{0}$ would be a general solution, something in parametric vector form, or a span of some vectors.

(c) False. The homogeneous equation *always* has the trivial solution. It has a *nontrivial* solution if and only if there's at least one free variable.

(d) False. It's backwards. What's true is that the equation $\mathbf{x} = \mathbf{p} + t\mathbf{v}$ describes a line through $\mathbf{p}$ parallel to $\mathbf{v}$. That's because $t\mathbf{v}$ is the line that passes through the origin and $\mathbf{v}$. When we add $\mathbf{p}$ to all its points, the line stays parallel to $\mathbf{v}$ but just gets shifted or "translated" by $\mathbf{p}$.

(e) True. (p. 52)

22. (a) False. For a nontrivial solution, some of its entries can be zero, we just need at least one that's nonzero.

(b) True. (p. 50)

(c) True! The author's really outdoing himself here. The given statement isn't the *definition* of "homogeneous," but it *is* logically true. If the zero vector is a solution of $A\mathbf{x} = \mathbf{b}$, then $A\mathbf{0} = \mathbf{b}$, right? But the left-hand side is $\mathbf{0}$. Therefore, $\mathbf{b} = \mathbf{0}$, and the equation is homogeneous after all.

We could rephrase this as a miniature theorem: *Theorem: The matrix equation $A\boldsymbol{x} = \boldsymbol{b}$ is homogeneous if and only if $\boldsymbol{0}$ is a solution.*

(d) True. (p. 51)

(e) True. (p. 51–52)