Homework Set #2 Solutions

Exercises 1.2 (p. 25)

Assignment: Do #5, 6, 7, 9, 11, 15, 17, 21, 25, 27, 29, 31, 33

5. (a) *Forward phase:*

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 6 & 7 & 8 & 7 \end{bmatrix} \xrightarrow{R2 \to R2 - 5R1} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 6 & 7 & 8 & 7 \end{bmatrix} \xrightarrow{R3 \to R3 - 6R1} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & -5 & -10 & -17 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & -5 & -10 & -17 \end{bmatrix} \xrightarrow{R3 \to R3 - \frac{5}{4}R2} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

Backward phase:

$$\begin{bmatrix} \boxed{1} & 2 & 3 & 4 \\ 0 & \boxed{-4} & -8 & -12 \\ 0 & 0 & 0 & -2 \end{bmatrix} \xrightarrow{R3 \to -\frac{1}{2}R3} \begin{bmatrix} \boxed{1} & 2 & 3 & 4 \\ 0 & \boxed{-4} & -8 & -12 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R1 \to R1 - 4R3} \begin{bmatrix} \boxed{1} & 2 & 3 & 0 \\ 0 & \boxed{-4} & -8 & -12 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R2 \to R2 + 12R3} \begin{bmatrix} \boxed{1} & 2 & 3 & 0 \\ 0 & \boxed{-4} & -8 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{array}{c}\begin{bmatrix} \boxed{1} & 2 & 3 & 0 \\ 0 & -4 & -8 & 0 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix} \\ \begin{array}{cccc} & \Uparrow & & \end{array}\end{array} \xrightarrow{R2 \to -\frac{1}{4}R2} \begin{array}{c}\begin{bmatrix} \boxed{1} & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix} \\ \begin{array}{cccc} & \Uparrow & & \end{array}\end{array} \xrightarrow{R1 \to R1 - 2R2} \begin{array}{c}\begin{bmatrix} \boxed{1} & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix} \\ \begin{array}{cccc} & \Uparrow & & \end{array}\end{array}$$

The final reduced echelon form with pivot positions and pivot columns identified is:

$$\begin{array}{c}\begin{bmatrix} \boxed{1} & 0 & -1 & 0 \\ 0 & \boxed{1} & 2 & 0 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix} \\ \begin{array}{cccc} \Uparrow & \Uparrow & & \Uparrow \end{array}\end{array}$$

(b) *Forward phase:*

The first and second rows are okay. It's only after we've covered the first two rows that we find something to do, since the pivot position doesn't contain a nonzero element:

$$\begin{array}{c}\begin{bmatrix} 1 & 3 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 \end{bmatrix} \\ \begin{array}{ccccc} & & & \Uparrow & \end{array}\end{array} \xrightarrow{R3 \leftrightarrow R4} \begin{array}{c}\begin{bmatrix} 1 & 3 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccccc} & & & \Uparrow & \end{array}\end{array}$$

That one ERO puts the matrix in echelon form.

Backward phase:

$$\begin{array}{c}\begin{bmatrix} \boxed{1} & 3 & 0 & 0 & 3 \\ 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccccc} & & & \Uparrow & \end{array}\end{array} \xrightarrow{R3 \to \frac{1}{3}R3} \begin{array}{c}\begin{bmatrix} \boxed{1} & 3 & 0 & 0 & 3 \\ 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccccc} & & & \Uparrow & \end{array}\end{array}$$

The final reduced echelon form with pivot positions and pivot columns identified is:

$$\begin{array}{c}\begin{bmatrix} \boxed{1} & 3 & 0 & 0 & 3 \\ 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccccc} \Uparrow & & \Uparrow & \Uparrow & \end{array}\end{array}$$

6. (a) *Forward phase:*

$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \\ 3 & 5 & 7 & 9 \end{bmatrix} \xrightarrow{R2 \to R2 - 2R1} \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & -2 & -4 & -6 \\ 3 & 5 & 7 & 9 \end{bmatrix} \xrightarrow{R3 \to R3 - 3R1} \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & -2 & -4 & -6 \\ 0 & -4 & -8 & -12 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & -2 & -4 & -6 \\ 0 & -4 & -8 & -12 \end{bmatrix} \xrightarrow{R3 \to R3 - 2R2} \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & -2 & -4 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Backward phase:

$$\begin{bmatrix} \boxed{1} & 3 & 5 & 7 \\ 0 & -2 & -4 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R2 \to -\frac{1}{2} R2} \begin{bmatrix} \boxed{1} & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R1 \to R1 - 3R2} \begin{bmatrix} \boxed{1} & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The final reduced echelon form with pivot positions and pivot columns identified is:

$$\begin{array}{c}\begin{bmatrix} \boxed{1} & 0 & -1 & -2 \\ 0 & \boxed{1} & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{cccc} \Uparrow & \Uparrow & & \end{array}\end{array}$$

(b) *Forward phase:*

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{R2 \leftrightarrow R3} \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Backward phase:

$$\begin{bmatrix} \boxed{1} & 0 & 1 & 0 & 0 \\ 0 & \boxed{1} & 0 & 1 & 0 \\ 0 & 0 & \boxed{1} & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{R2 \to R2 - R4} \begin{bmatrix} \boxed{1} & 0 & 1 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 & -1 \\ 0 & 0 & \boxed{1} & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \boxed{1} & 0 & 1 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & \boxed{1} & 1 \end{bmatrix} \xrightarrow{R1 \to R1 - R3} \begin{bmatrix} \boxed{1} & 0 & 0 & 0 & -1 \\ 0 & \boxed{1} & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & \boxed{1} & 1 \end{bmatrix}$$

The final reduced echelon form with pivot positions and pivot columns identified is:

$$\begin{bmatrix} \boxed{1} & 0 & 0 & 0 & -1 \\ 0 & \boxed{1} & 0 & 0 & -1 \\ 0 & 0 & \boxed{1} & 0 & 1 \\ 0 & 0 & 0 & \boxed{1} & 1 \end{bmatrix}$$

7. *Forward phase:*

$$\begin{bmatrix} 1 & 0 & 2 & 5 \\ 2 & 0 & 3 & 6 \end{bmatrix} \xrightarrow{R2 \to R2 - 2R1} \begin{bmatrix} 1 & 0 & 2 & 5 \\ 0 & 0 & -1 & -4 \end{bmatrix}$$

Backward phase:

$$\begin{bmatrix} \boxed{1} & 0 & 2 & 5 \\ 0 & 0 & -1 & -4 \end{bmatrix} \xrightarrow{R2 \to -R2} \begin{bmatrix} \boxed{1} & 0 & 2 & 5 \\ 0 & 0 & 1 & 4 \end{bmatrix} \xrightarrow{R1 \to R1 - 2R2} \begin{bmatrix} \boxed{1} & 0 & 0 & -3 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

This corresponds to the general solution:

$$\begin{cases} x_1 = -3 \\ x_3 = 4 \\ x_2 \text{ free} \end{cases}$$

9. *Forward phase:*

$$\begin{bmatrix} 0 & 3 & 6 & 9 \\ -1 & 1 & -2 & -1 \end{bmatrix} \xrightarrow{R1 \leftrightarrow R2} \begin{bmatrix} -1 & 1 & -2 & -1 \\ 0 & 3 & 6 & 9 \end{bmatrix}$$

Backward phase:

$$\begin{bmatrix} \boxed{-1} & 1 & -2 & -1 \\ 0 & 3 & 6 & 9 \end{bmatrix} \xrightarrow{R2 \to \frac{1}{3}R2} \begin{bmatrix} \boxed{-1} & 1 & -2 & -1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \xrightarrow{R1 \to R1 - R2} \begin{bmatrix} \boxed{-1} & 0 & -4 & -4 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & -4 & -4 \\ 0 & \boxed{1} & 2 & 3 \end{bmatrix} \xrightarrow{R1 \to -R1} \begin{bmatrix} 1 & 0 & 4 & 4 \\ 0 & \boxed{1} & 2 & 3 \end{bmatrix}$$

This corresponds to the general solution:

$$\begin{cases} x_1 = 4 - 4x_3 \\ x_2 = 3 - 2x_3 \\ x_3 \text{ free} \end{cases}$$

11. *Forward phase:*

$$\begin{bmatrix} 1 & 2 & -7 \\ -1 & -1 & 1 \\ 2 & 1 & 5 \end{bmatrix} \xrightarrow{R2 \to R2+R1} \begin{bmatrix} 1 & 2 & -7 \\ 0 & 1 & -6 \\ 2 & 1 & 5 \end{bmatrix} \xrightarrow{R3 \to R3-2R1} \begin{bmatrix} 1 & 2 & -7 \\ 0 & 1 & -6 \\ 0 & -3 & 19 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -7 \\ 0 & 1 & -6 \\ 0 & -3 & 19 \end{bmatrix} \xrightarrow{R3 \to R3+3R2} \begin{bmatrix} 1 & 2 & -7 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix}$$

This echelon form contains a row of the form $[0\,0\ldots b]$ for $b \neq 0$. Therefore, the system is inconsistent, and the solution set is empty.

15. *Forward phase:*

The matrix is already in echelon form.

Backward phase:

$$\begin{bmatrix} \boxed{1} & -2 & 0 & 0 & 7 & -3 \\ 0 & 1 & 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & \boxed{1} & 5 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R1 \to R1+2R2} \begin{bmatrix} \boxed{1} & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & \boxed{1} & 5 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This corresponds to the general solution:

$$\begin{cases} x_1 = -1 - x_5 \\ x_2 = 1 + 3x_5 \\ x_4 = -4 - 5x_5 \\ x_3,\ x_5 \text{ free} \end{cases}$$

17.

$$\begin{bmatrix} 1 & 4 & 2 \\ -3 & h & -1 \end{bmatrix} \xrightarrow{R2 \to R2+3R1} \begin{bmatrix} 1 & 4 & 2 \\ 0 & h+12 & 5 \end{bmatrix}$$

Observe that the underlying system is consistent if and only if $h \neq -12$.

21.

$$\begin{bmatrix} 1 & h & 1 \\ 2 & 3 & k \end{bmatrix} \xrightarrow{R2 \to R2 - 2R1} \begin{bmatrix} 1 & h & 1 \\ 0 & 3-2h & k-2 \end{bmatrix}$$
$$\Uparrow \qquad\qquad\qquad\qquad \Uparrow$$

Observe that there are three possible forms of the last row:

(a) $[0\ 0\ b]$ with $b \neq 0$ if we choose $h = 3/2$ and any $k \neq 2$: in this case, the system is inconsistent and has *no solutions*.

(b) $[0\ a\ b]$ with $a \neq 0$ (and b zero or nonzero) if we choose $h \neq 3/2$ and any k whatsoever: in this case, the system has a *unique solution*.

(a) $[0\ 0\ 0]$ (if $h = 3/2$ and $k = 2$): in this case, x_2 is a free variable and the system has *infinitely many solutions*.

25. This problem is relying on the following fact that I discussed in *Homework Set #1 Solutions* Exercise 1.2–25(c). If we were to separately find the pivot positions of the coefficient matrix and those of the augmented matrix, we'd discover that the pivot positions of the augmented matrix would include all those of the coefficient matrix plus *maybe* one more—an extra pivot position in the rightmost column that we added to form the augmented matrix.

In this problem, we're told that a 3×5 coefficient matrix has three pivot columns. That means it has three pivot *positions*, but each pivot position occurs in a different row. Therefore, the coefficient matrix contains a pivot position in *every* row.

Now, when we look at the augmented matrix, it has the same three pivot positions, one in each row. It *can't* have another pivot position in the rightmost column, because it doesn't have an extra row where it could fit!

Therefore, the augmented matrix's rightmost column is *not* a pivot column, so the associated system is consistent.

If you got that question right, you really know this stuff well!

27. This problem is really the same thing as problem 25. Again, if the coefficient matrix has a pivot position in every row, then the augmented matrix can't have an "extra" pivot position: it just has exactly the same pivot positions as the coefficient matrix. None of these pivot positions can be in the augmented matrix's rightmost column (because that's the one we added, and the pivot positions are all in the coefficient columns). Therefore, the system is consistent.

29. *"If a linear system is consistent, then the solution is unique if and only if every column of the coefficient matrix is a pivot column."* OR *"If a linear system is consistent, then the solution is unique if and only if every column of the augmented matrix except the rightmost column is a pivot column."*

31. If a system of linear equations has fewer equations than unknowns, then its coefficient matrix has fewer rows than columns. The number of pivot positions can't be more than the number of rows, so the coefficient matrix has fewer pivot columns than total columns. That is, not every column is a pivot column. By the answer to problem 29, the solution of the associated system isn't unique; there must be an infinite number of solutions.

33. Yes. Here's a simple example:

$$\begin{cases} x + y = 10 \\ x + y = 10 \\ x + y = 10 \end{cases}$$

Even though it's the same equation used three times, it's still a system of three equations in two unknowns. It's consistent because any two numbers that add to ten can be used as a solution.

Exercises 1.3 (p. 36)

Assignment: Do #31, 32, 1, 3, 5, 7, 11

31. (a) False. Another notation for $\begin{bmatrix} -4 \\ 3 \end{bmatrix}$ is $(-4, 3)$ which looks a little like $\begin{bmatrix} -4 & 3 \end{bmatrix}$. However, as discussed in class, these two matrices can't be equal because they are different sizes. One is 2×1 and the other is 1×2.

(b) False. The author is asking if these two points lie on the *same* line through the origin. They don't, as you can see by plotting them or by noting that $(-2, 5)$ is not a scalar multiple of $(-5, 2)$.

(c) True. (p. 31)

(d) True. (p. 34)

(e) False. This is a clever question. The span of two vectors can *usually* be visualized as a plane through the origin. But, if one vector is a scalar multiple of the other, their span is a *line* through the origin. If both vectors are zero, their span is the *point* at the origin.

32. (a) True. (p. 31)

(b) True. $(\mathbf{u}-\mathbf{v})+\mathbf{v} = \mathbf{u}+(-\mathbf{v}+\mathbf{v}) = \mathbf{u}+\mathbf{0} = \mathbf{u}$ by applying various parts of the *Alebraic Properties of* $\mathbb{R}^n$ on p. 31.

(c) False. (p. 31) If they are all zero, we just get a linear combination equal to $\mathbf{0}$.

(d) True. (p. 35)

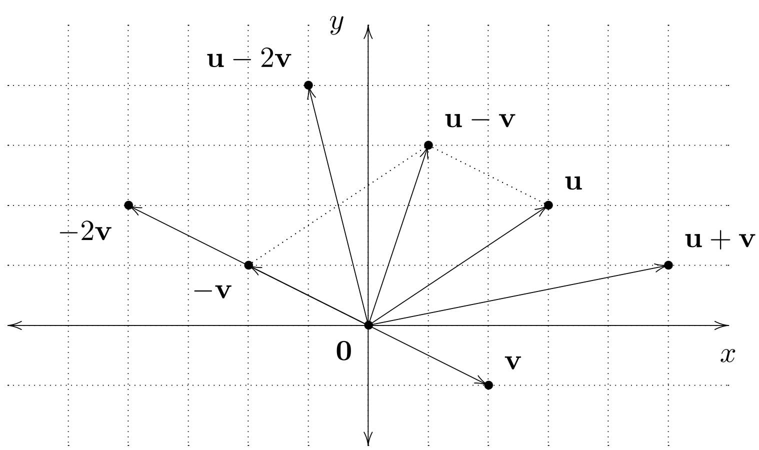
(e) True. (p. 34)

1.

$$\mathbf{u}+\mathbf{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

$$\mathbf{u}-2\mathbf{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} - 2\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} - \begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

3. In addition to $\mathbf{u} = (3,2)$ and $\mathbf{v} = (2,-1)$, we have $-\mathbf{v} = (-2,1)$, $-2\mathbf{v} = (-4,2)$, $\mathbf{u}+\mathbf{v} = (5,1)$, $\mathbf{u}-\mathbf{v} = (1,3)$, and $\mathbf{u}-2\mathbf{v} = (-1,4)$. Plotting these, we have:



5. $\begin{cases} 3x_1 - 2x_2 = 8 \\ x_1 = -6 \\ -5x_1 + 4x_2 = 3 \end{cases}$

7. $\mathbf{a} = 2\mathbf{u} - \mathbf{v}$
$\mathbf{b} = 2\mathbf{u} - 2\mathbf{v}$
$\mathbf{c} = 3.5\mathbf{u} - 2\mathbf{v}$
$\mathbf{d} = 4\mathbf{u} - 3\mathbf{v}$

Yes, all vectors in $\mathbb{R}^2$ are linear combinations of $\mathbf{u}$ and $\mathbf{v}$.

11. This is equivalent to whether the linear system corresponding to the augmented matrix

$$\begin{bmatrix} 1 & -2 & -6 & 11 \\ 0 & 3 & 7 & -5 \\ 1 & -2 & 5 & 9 \end{bmatrix}$$

has a solution. Reducing this matrix to echelon form:

$$\begin{array}{c} \begin{bmatrix} 1 & -2 & -6 & 11 \\ 0 & 3 & 7 & -5 \\ 1 & -2 & 5 & 9 \end{bmatrix} \\ \begin{array}{llll} \Uparrow & & & \end{array} \end{array} \xrightarrow{R3 \to R3 - R1} \begin{array}{c} \begin{bmatrix} 1 & -2 & -6 & 11 \\ 0 & 3 & 7 & -5 \\ 0 & 0 & 11 & -2 \end{bmatrix} \\ \begin{array}{llll} \Uparrow & & & \end{array} \end{array}$$

We see that the associated system is consistent. Therefore, *yes*, $\mathbf{b}$ is a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$.

Exercises 1.4 (p. 46)

Assignment: Do #29, 30

29. (a) False. (p. 40) It is a *matrix equation.*

(b) True. (p. 41)

(c) False. It is possible for the system to be consistent or inconsistent: it all depends on whether the rightmost column has a pivot position in it.

(d) True. (p. 43)

(e) True. (p. 42, Theorem 4)

(f) True. (p. 42, Theorem 4)

30. (a) False. This text reserves the term *vector equation* for an equation of the form

$$x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

not just any equation that might involve vectors. For example, the equation $A\mathbf{x} = \mathbf{b}$ involves vectors, but its called a *matrix equation.*

(b) True. (p. 39)

(c) True. (p. 41)

(d) True. (p. 41)

(e) False. It is possible for the equation to be consistent or inconsistent: again, it all depends on whether the rightmost column has a pivot position or not.

(f) True. (p. 42, Theorem 4)