Homework Set #1 Solutions

Exercises 1.1 (p. 10)

Assignment: Do #33, 34, 1, 3, 5, 29-31, 17, 19, 21, 23, 25, 27

33. (a) True. (p. 7)

(b) False. It has five rows and six columns.

(c) False. The definition given is that for a single *solution*. In contrast, the *solution set* is the set of all such solutions.

(d) True. (p. 8)

34. (a) True. (p. 7)

(b) False. Two matrices are row equivalent if one can be transformed to the other by means of elementary row operations.

(c) False. An inconsistent system has no solutions.

(d) True. (p. 3)

For problems 1, 3, and 5, I've used the row reduction algorithm of Section 1.2 on the augmented matrices to solve the systems. If you picked different row operations, your intermediate steps will be different, but you should get the same final solution set as I do.

Here, I've solved problem 1 in unusual detail. Problems 3 and 5 are solved in the detail I'd expect on a quiz or exam if I asked you to show your work.

1. We begin by locating the pivot column and the pivot position (step 1):

$$\begin{bmatrix} \textcircled{1} & 7 & 4 \\ -2 & -9 & 2 \end{bmatrix}$$
$$\Uparrow$$

Since the pivot position contains a nonzero pivot, there's no need to interchange rows (step 2). It now remains to zero out the entry below the pivot (step 3), which we do here:

$$\begin{bmatrix} \textcircled{1} & 7 & 4 \\ -2 & -9 & 2 \end{bmatrix} \xrightarrow{R2 \to R2+2R1} \begin{bmatrix} \textcircled{1} & 7 & 4 \\ 0 & 5 & 10 \end{bmatrix}$$
$$\Uparrow \qquad\qquad\qquad\qquad \Uparrow$$

Now, we "cover" the pivot row, and repeat the steps for the submatrix (step 4).

We identify the new pivot column and pivot position (step 1) as follows:

$$\begin{bmatrix} 1 & 7 & 4 \\ 0 & \textcircled{5} & 10 \end{bmatrix}$$
$$\qquad\Uparrow$$

As the pivot position contains a nonzero entry, there's nothing to do for step 2. As there are no entries below the pivot, there's nothing to do for step 3. Therefore, we are done the forward phase of the row reduction algorithm.

Observe that now the matrix is in an echelon form. From this form, we can already apply Theorem 2 on p. 24 to see that (i) the original linear system is consistent (because the rightmost column of the augmented matrix isn't a pivot column); and (ii) it has a unique solution (because all variables correspond to pivot columns and so are basic variables—there are no free variables).

Now we can apply the backwards phase (step 5) to produce the reduced echelon form. Working from right to left, we attack each of the pivots. The first pivot is 5, and it must be scaled to one:

$$\begin{bmatrix} \boxed{1} & 7 & 4 \\ 0 & \textcircled{5} & 10 \end{bmatrix} \xrightarrow{R2 \to \frac{1}{5}R2} \begin{bmatrix} \boxed{1} & 7 & 4 \\ 0 & \textcircled{1} & 2 \end{bmatrix}$$

Now, it's necessary to "zero out" the entry above this pivot:

$$\begin{bmatrix} \boxed{1} & 7 & 4 \\ 0 & \textcircled{1} & 2 \end{bmatrix} \xrightarrow{R1 \to R1 - 7R2} \begin{bmatrix} \boxed{1} & 0 & -10 \\ 0 & \textcircled{1} & 2 \end{bmatrix}$$

That's done, so we can move on to the next pivot:

$$\begin{bmatrix} \textcircled{1} & 0 & -10 \\ 0 & \boxed{1} & 2 \end{bmatrix}$$
$$\Uparrow$$

This pivot is already 1, and there are no entries above it to "zero out." Therefore, we are done. Observe that the final matrix is in reduced echelon form, and it corresponds to the general solution

$$\begin{cases} x_1 = -10 \\ x_2 = 2 \end{cases}$$

with basic variables x_1 and x_2 and no free variables. In short, $(-10, 2)$ is the unique solution.

It is always worthwhile to check our answer, since it's easy to make a small mistake while doing a row operation. Substituting this answer into the original equations gives:

$$\begin{aligned} x_1 + 7x_2 &= -10 + 7(2) = 4 \\ -2x_1 - 9x_2 &= -2(-10) - 9(2) = 2 \end{aligned}$$

and we see that we've gotten it right.

3. Identify first pivot position. It's already got a nonzero pivot, so just zero out the entry below:

$$\begin{bmatrix} \textcircled{1} & -3 & 4 \\ -3 & 9 & 8 \end{bmatrix} \xrightarrow{R2 \to R2+3R1} \begin{bmatrix} \textcircled{1} & -3 & 4 \\ 0 & 0 & 20 \end{bmatrix}$$
$$\Uparrow \qquad\qquad\qquad\qquad \Uparrow$$

Cover first row, and identify next pivot position:

$$\begin{bmatrix} 1 & -3 & 4 \\ 0 & 0 & \textcircled{20} \end{bmatrix}$$
$$\Uparrow$$

There's nothing else to do!

Now, the matrix is in echelon form. Already, we can see that it is inconsistent because the last row is equivalent to the invalid equation $0 = 20$. There is no need to continue; we know the solution set is empty.

5. Identify first pivot position. It's already got a nonzero pivot, so just zero out the entry below:

$$\begin{bmatrix} \textcircled{1} & 4 & 7 \\ 1 & -1 & -1 \end{bmatrix} \xrightarrow{R2 \to R2-R1} \begin{bmatrix} \textcircled{1} & 4 & 7 \\ 0 & -5 & -8 \end{bmatrix}$$
$$\Uparrow \qquad\qquad\qquad\qquad \Uparrow$$

Cover first row, and identify next pivot position:

$$\begin{bmatrix} 1 & 4 & 7 \\ 0 & \textcircled{-5} & -8 \end{bmatrix}$$
$$\Uparrow \text{ (column 2)}$$

There's nothing else to do!

The echelon form tells us the underlying linear system is consistent (last column isn't a pivot column) and has a unique solution (all other columns are pivot columns). We continue with the backward phase to get the solution.

Working right to left, we find our first pivot must be scaled to 1. Then, the entry above it must be "zeroed out":

$$\begin{bmatrix} \boxed{1} & 4 & 7 \\ 0 & \textcircled{-5} & -8 \end{bmatrix} \xrightarrow{R2 \to -\frac{1}{5}R2} \begin{bmatrix} \boxed{1} & 4 & 7 \\ 0 & \textcircled{1} & \frac{8}{5} \end{bmatrix} \xrightarrow{R1 \to R1 - 4R2} \begin{bmatrix} \boxed{1} & 0 & \frac{3}{5} \\ 0 & \textcircled{1} & \frac{8}{5} \end{bmatrix}$$
$$\Uparrow \text{ (column 2 in each)}$$

The next pivot is already one, and there are no entries above it:

$$\begin{bmatrix} \textcircled{1} & 0 & \frac{3}{5} \\ 0 & \boxed{1} & \frac{8}{5} \end{bmatrix}$$
$$\Uparrow \text{ (column 1)}$$

This is the reduced echelon form, giving the unique solution:

$$\begin{cases} x_1 = \frac{3}{5} \\ x_2 = \frac{8}{5} \end{cases}$$

and, again, it's worthwhile to check our solution against the original system:

$$\begin{aligned} x_1 + 4x_2 &= \tfrac{3}{5} - 4\left(\tfrac{8}{5}\right) = 7 \\ x_1 - x_2 &= \tfrac{3}{5} - \tfrac{8}{5} = -1 \end{aligned}$$

Therefore, the point $(\frac{3}{5}, \frac{8}{5})$ is the unique point lying on both given lines.

29. forward: $R2 \to \frac{1}{2}R2$; reverse $R2 \to 2R2$.

30. forward: $R1 \leftrightarrow R2$; reverse $R1 \leftrightarrow R2$.

31. forward: $R3 \to R3 - 2R2$; reverse $R3 \to R3 + 2R2$.

For problems 17 and 19, I've used the forward phase of the row reduction algorithm on the augmented matrices to get an echelon form. This tells me where the pivot columns are, and I can apply Theorem 2 on p. 24: the systems are consistent if and only if the rightmost column isn't a pivot column.

17. We find the first pivot position. It's already got a nonzero pivot, so we need only zero out the entries below it:

$$\begin{bmatrix} \textcircled{-2} & -3 & 4 & 5 \\ 0 & 1 & -2 & 4 \\ 1 & 3 & -1 & 2 \end{bmatrix} \xrightarrow{R3\to R3+\frac{1}{2}R1} \begin{bmatrix} \textcircled{-2} & -3 & 4 & 5 \\ 0 & 1 & -2 & 4 \\ 0 & \frac{3}{2} & 1 & \frac{9}{2} \end{bmatrix}$$
$$\Uparrow \qquad\qquad\qquad\qquad \Uparrow$$

We cover the first row, and identify the next pivot position. It already contains a nonzero pivot which we use to zero out the entry below it:

$$\begin{bmatrix} -2 & -3 & 4 & 5 \\ 0 & \textcircled{1} & -2 & 4 \\ 0 & \frac{3}{2} & 1 & \frac{9}{2} \end{bmatrix} \xrightarrow{R3\to R3-\frac{3}{2}R2} \begin{bmatrix} -2 & -3 & 4 & 5 \\ 0 & \textcircled{1} & -2 & 4 \\ 0 & 0 & 4 & -\frac{3}{2} \end{bmatrix}$$
$$\Uparrow \qquad\qquad\qquad\qquad \Uparrow$$

We cover the first and second rows, and identify the next pivot position:

$$\begin{bmatrix} -2 & -3 & 4 & 5 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & \textcircled{4} & -\frac{3}{2} \end{bmatrix}$$
$$\Uparrow$$

Now, there's nothing else to do. Our matrix is in echelon form. If we indicate all pivot columns and pivot positions:

$$\begin{bmatrix} \boxed{-2} & -3 & 4 & 5 \\ 0 & \boxed{1} & -2 & 4 \\ 0 & 0 & \boxed{4} & -\frac{3}{2} \end{bmatrix}$$
$$\begin{matrix} \Uparrow & \Uparrow & \Uparrow & \phantom{-\frac{3}{2}} \end{matrix}$$

we see that the rightmost column is not a pivot column. Therefore, by Theorem 2, this system is consistent. (In fact, since all *other* columns are pivot columns, it has no free variables and so has a unique solution.)

19. We find the first pivot position but discover that it contains a zero entry. We need to interchange two rows to get a nonzero pivot:

$$\begin{bmatrix} \textcircled{0} & 2 & 2 & 0 & 0 \\ 1 & 0 & 0 & -2 & -3 \\ 0 & 0 & 1 & 3 & -4 \\ -2 & 3 & 2 & 1 & 5 \end{bmatrix} \xrightarrow{R1 \leftrightarrow R2} \begin{bmatrix} \textcircled{1} & 0 & 0 & -2 & -3 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & -4 \\ -2 & 3 & 2 & 1 & 5 \end{bmatrix}$$
$$\begin{matrix} \Uparrow & & & & & & \Uparrow \end{matrix}$$

Now, we can use this pivot to zero out the nonzero entry below:

$$\xrightarrow{R4 \to R4 + 2R1} \begin{bmatrix} \textcircled{1} & 0 & 0 & -2 & -3 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & -4 \\ 0 & 3 & 2 & -3 & -1 \end{bmatrix}$$
$$\begin{matrix} \Uparrow \end{matrix}$$

We cover the first row and identify the next pivot position. Fortunately, it's nonzero, so we need only zero out the entries below it:

$$\begin{bmatrix} 1 & 0 & 0 & -2 & -3 \\ 0 & \textcircled{2} & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & -4 \\ 0 & 3 & 2 & -3 & -1 \end{bmatrix} \xrightarrow{R4 \to R4 - \frac{3}{2}R2} \begin{bmatrix} 1 & 0 & 0 & -2 & -3 \\ 0 & \textcircled{2} & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & -4 \\ 0 & 0 & -1 & -3 & -1 \end{bmatrix}$$
$$\begin{matrix} & \Uparrow & & & & & & \Uparrow \end{matrix}$$

We cover the first and second rows, and identify the next pivot position. It, too, is nonzero, and we need only zero out the entry below it:

$$\begin{bmatrix} 1 & 0 & 0 & -2 & -3 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & \textcircled{1} & 3 & -4 \\ 0 & 0 & -1 & -3 & -1 \end{bmatrix} \xrightarrow{R4 \to R4 + R3} \begin{bmatrix} 1 & 0 & 0 & -2 & -3 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & \textcircled{1} & 3 & -4 \\ 0 & 0 & 0 & 0 & -5 \end{bmatrix}$$

$$\Uparrow \qquad\qquad\qquad \Uparrow$$

If we cover up the first three rows and identify the pivot position:

$$\begin{bmatrix} 1 & 0 & 0 & -2 & -3 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & -4 \\ 0 & 0 & 0 & 0 & \textcircled{-5} \end{bmatrix}$$

$$\Uparrow$$

we see that there's nothing else to do.

This final matrix is in echelon form. What's more, in the last step, we identified the rightmost column as a pivot column. By Theorem 2, this system is inconsistent. It has no solutions.

The trick with problems 21, 23, and 25 is to apply the row reduction algorithm just as if we knew the value for the unspecified matrix entry (or entries).

21. The first pivot position contains a nonzero entry. We just need to zero out the entry below it:

$$\begin{bmatrix} \textcircled{1} & -3 & h \\ -2 & 6 & -5 \end{bmatrix} \xrightarrow{R2 \to R2 + 2R1} \begin{bmatrix} \textcircled{1} & -3 & h \\ 0 & 0 & -5+2h \end{bmatrix}$$

$$\Uparrow \qquad\qquad\qquad \Uparrow$$

At this stage, we cover the first row and try to find the next pivot position:

$$\begin{bmatrix} 1 & -3 & h \\ 0 & 0 & -5+2h \end{bmatrix}$$

Now, the pivot column will be the leftmost nonzero column. Either the unknown quantity $-5 + 2h$ is zero or it's nonzero. We need to consider these two cases separately.

Case 1. If $-5 + 2h = 0$, then there *is* no leftmost nonzero column, so the algorithm stops. In this case, the matrix is in echelon form, and the pivot columns and positions are as follows:

$$\begin{bmatrix} \boxed{1} & -3 & h \\ 0 & 0 & 0 \end{bmatrix}$$
$$\Uparrow$$

Since the rightmost column is *not* a pivot column, in this case the associated system is consistent.

Case 2. If $-5 + 2h \neq 0$, then there *is* a leftmost nonzero column, and we identify it as the pivot column. The nonzero quantity $-5 + 2h$ is our pivot, it's already nonzero, and there are no entries below it to zero out. Therefore, we are done, and our matrix is in echelon form with pivot columns and positions as follows:

$$\begin{bmatrix} \boxed{1} & -3 & h \\ 0 & 0 & \boxed{-5+2h} \end{bmatrix}$$
$$\Uparrow \qquad\qquad \Uparrow$$

Since the rightmost column *is* a pivot column, in this case the associated system is inconsistent.

Therefore, our associated system is consistent if and only if $-5 + 2h = 0$ or, equivalently, if and only if $h = 5/2$.

23. The first pivot position contains a nonzero entry, and there's a nonzero entry below it that must be zeroed out:

$$\begin{bmatrix} \textcircled{1} & h & -2 \\ -4 & 2 & 10 \end{bmatrix} \xrightarrow{R2 \to R2+4R1} \begin{bmatrix} \textcircled{1} & h & -2 \\ 0 & 2+4h & 2 \end{bmatrix}$$
$$\Uparrow \qquad\qquad\qquad\qquad \Uparrow$$

Now, we cover the first row and try to identify the next pivot column:

$$\begin{bmatrix} 1 & h & -2 \\ 0 & 2+4h & 2 \end{bmatrix}$$

Again, this depends on $2 + 4h$, and we must study the two cases:

Case 1. If $2 + 4h = 0$, then the next pivot column the rightmost column. Immediately, we know the associated linear system must be inconsistent.

Case 2. On the other hand, if $2 + 4h \neq 0$, then the next pivot column is column two. In this case, the pivot position has entry $2 + 4h$ (which is nonzero) and no entries below it, so we are done. The matrix is in echelon form, and its pivot columns are the first and second columns. Since its rightmost column is *not* a pivot column, the associated linear system is consistent.

Therefore, the associated system is consistent if and only if $2+4h \neq 0$ or, equivalently, if and only if $h \neq -1/2$.

25. This exercise is another variant on the theme.

The first pivot position is easy to identify. It contains a nonzero pivot, and we just have to zero out a nonzero entry below it:

$$\begin{bmatrix} \textcircled{1} & -4 & 7 & g \\ 0 & 3 & -5 & h \\ -2 & 5 & -9 & k \end{bmatrix} \xrightarrow{R3 \to R3+2R1} \begin{bmatrix} \textcircled{1} & -4 & 7 & g \\ 0 & 3 & -5 & h \\ 0 & -3 & 5 & 2g+k \end{bmatrix}$$
$$\Uparrow \qquad\qquad\qquad\qquad \Uparrow$$

Moving on, the next pivot position also contains a nonzero pivot, and we just have to zero out the nonzero entry below it:

$$\begin{bmatrix} 1 & -4 & 7 & g \\ 0 & \textcircled{3} & -5 & h \\ 0 & -3 & 5 & 2g+k \end{bmatrix} \xrightarrow{R3 \to R3+R2} \begin{bmatrix} 1 & -4 & 7 & g \\ 0 & \textcircled{3} & -5 & h \\ 0 & 0 & 0 & 2g+h+k \end{bmatrix}$$
$$\Uparrow \qquad\qquad\qquad\qquad \Uparrow$$

Now, as before, the location of the final pivot column depends on the value of $2g + h + k$. If $2g + h + k = 0$, then there is no additional pivot column; the only two pivot columns are the first and second columns, the rightmost column is not a pivot column, and the associated system is consistent. On the other hand, if $2g + h + k \neq 0$, then the final pivot column is the rightmost column, and the associated system is inconsistent.

Therefore, the associated linear system is consistent if and only if $2g + h + k = 0$.

27. We apply the row reduction algorithm to the augmented matrix associated with this linear system.

$$\begin{bmatrix} \textcircled{2} & 3 & -1 \\ 6 & 5 & 0 \\ 2 & -5 & 7 \end{bmatrix} \xrightarrow{R2 \to R2 - 3R1} \begin{bmatrix} \textcircled{2} & 3 & -1 \\ 0 & -4 & 3 \\ 2 & -5 & 7 \end{bmatrix} \xrightarrow{R3 \to R3 - R1} \begin{bmatrix} \textcircled{2} & 3 & -1 \\ 0 & -4 & 3 \\ 0 & -8 & 8 \end{bmatrix}$$

$$\Uparrow \qquad\qquad\qquad \Uparrow \qquad\qquad\qquad \Uparrow$$

$$\begin{bmatrix} 2 & 3 & -1 \\ 0 & \textcircled{-4} & 3 \\ 0 & -8 & 8 \end{bmatrix} \xrightarrow{R3 \to R3 - 2R2} \begin{bmatrix} 2 & 3 & -1 \\ 0 & \textcircled{-4} & 3 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\Uparrow \qquad\qquad\qquad \Uparrow$$

Now, the matrix is in echelon form. Since this system is inconsistent (observe that the rightmost column is a pivot column), the lines have no point in common.

Exercises 1.2 (p. 25)

Assignment: Do #23, 24, 1, 3

23. (a) False. A matrix may have different echelon forms, but its *reduced* echelon form is unique.

(b) False. The row reduction algorithm may be applied to any matrix to reduce it to echelon or reduced echelon form. The matrix doesn't *have* to be the augmented matrix of a linear system.

(c) True, but... On page 20, the author defines *basic variables* as those variables corresponding to pivot columns of the *augmented* matrix, not the coefficient matrix. However, the pivot columns of the coefficient matrix (what we'd get if we used our row reduction algorithm on the coefficient matrix directly) are pivot columns of the augmented matrix, too. (Can you see why?) Therefore, the basic variables do in fact correspond to pivot columns of the coefficient matrix. I don't think the author meant to make this problem so hard.

(d) Mostly true. (p. 22) Finding a parameteric description *or showing that the solution set is empty* is the same as solving the system.

(e) False. If the augmented matrix contains a row that's all zeros except for a nonzero *rightmost* entry, then the system is inconsistent. That's because such a row corresponds to an equation $0 = b$ (for b nonzero) which is clearly contradictory. The row given here corresponds to the equation $x_4 = 0$, and there's nothing wrong with that!

24. (a) False. See 23(a).

(b) False. The pivot positions of a matrix are an intrinsic property of that matrix. They don't depend on what kind of row reduction technique we apply to it.

(c) True. (p. 17, 20)

(d) True. (p. 24)

(e) True. (p. 21)

(f) False. The ostrich is the largest bird living *today*, but both the Elephant Bird of Madagascar and some species of New Zealand Moas were *much* bigger.

1. Matrix (d) isn't in echelon form: it has a zero row at the top. The rest are in echelon form.

Matrices (a) and (b) are also in reduced echelon form. Matrix (c) isn't in reduced echelon form because of its third column (the middle entry is a leading entry with a nonzero entry above it).

3. Both are in echelon form. Matrix (a) is in reduced echelon form, too. Note that the "1" in row 1, column 3 is okay because the "1" in row 2, column 3 *isn't* a leading entry. When in doubt, circle the leading entries.