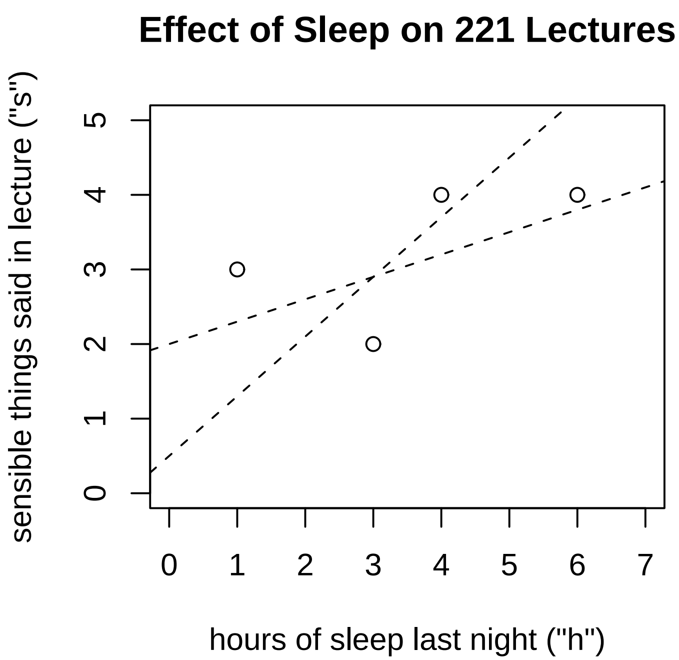
Some Silly Data



Try to explain relationship by "fitting" a line.

Fitting a Line

$h =$ hours of sleep last night
$s =$ sensible things said in lecture

Four lectures' worth of data:

i	h_i	s_i
1	1	3
2	3	2
3	4	4
4	6	4

Fitting to a line means "solving":

$$\begin{aligned} s_1 &= ah_1 + b & 3 &= 1a + b \\ s_2 &= ah_2 + b & 2 &= 3a + b \\ s_3 &= ah_3 + b & 4 &= 4a + b \\ s_4 &= ah_4 + b & 4 &= 6a + b \end{aligned}$$

for a the slope, b the intercept. In other words,

$$\underbrace{\mathbf{s}}_{4\times 1} = \underbrace{\begin{bmatrix} \mathbf{h} & \mathbf{1} \end{bmatrix}}_{4\times 2} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{2\times 1}$$

Least-Squares Line

We can't expect to *solve*

$$\mathbf{s} = \underbrace{\begin{bmatrix} \mathbf{h} & \mathbf{1} \end{bmatrix}}_{H} \begin{bmatrix} a \\ b \end{bmatrix}$$

exactly, but we can find the solution $\begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix}$ that best approximates **s** by a linear combination of **h** and **1**.

Solving:

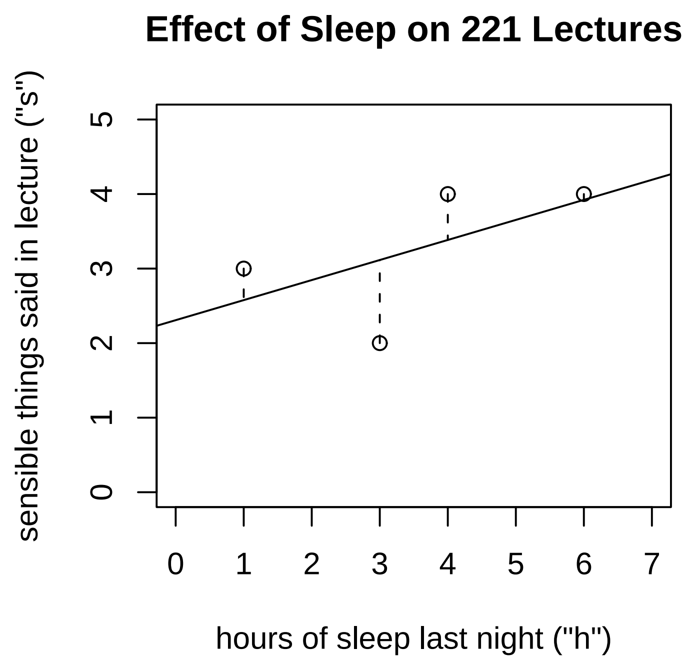
$$H^T H \begin{bmatrix} a \\ b \end{bmatrix} = H^T \mathbf{s}$$

$$\begin{bmatrix} 62 & 14 \\ 14 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 49 \\ 13 \end{bmatrix}$$

has the unique solution

$$\begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix} = \begin{bmatrix} 0.269 \\ 2.308 \end{bmatrix}$$

Least-Squares Line



The least-squares line minimizes the sum of squared errors (the sum of squared lengths of the dashed lines).

Orthogonal Diagonalization

Let A be symmetric, $n \times n$.

1. Find eigenvalues.

2. Find bases for eigenspaces. (You must have n vectors in total, or you made a mistake.)

3. For each basis with more than one vector, use Gram-Schmidt to orthogonalize it.

4. Take all your orthogonal vectors and normalize them.

5. (a) Take these n orthonormal vectors as the columns of P;
   (b) Create D from the eigenvalues (in *same order*);
   (c) Calculate $P^{-1} = P^T$.

6. Now, $A = PDP^{-1} = PDP^T$.

Spectral Theorem

Definition: The set of eigenvalues of a matrix is its *spectrum*.

Theorem: An $n \times n$, symmetric matrix:

(a) has n real eigenvalues counting multiplicities;

(b) has the dimension of the eigenspace for each eigenvalue equal to its multiplicity;

(c) has mutually orthogonal eigenspaces (that is, vectors from distinct eigenspaces are orthogonal);

(d) is orthogonally diagonalizable.