Pet Store Puzzle

Pet store owner orders assistant to take inventory of dogs and birds. Assistant returns and proudly announces there are 8 heads and 22 feet in total.

How many dogs and birds are there?

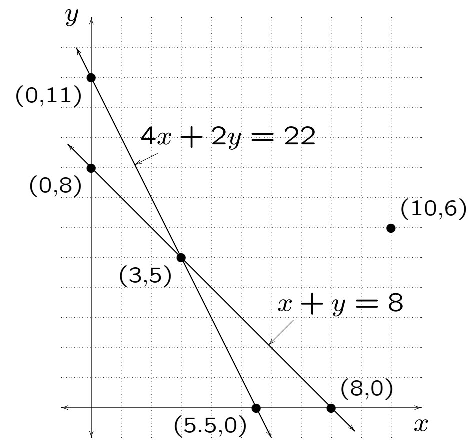
Let x be the number of dogs.
Let y be the number of birds.

$$\begin{cases} x + y = 8 & \text{(heads)} \\ 4x + 2y = 22 & \text{(feet)} \end{cases}$$

These are two *simultaneous equations* in two *variables* or *unknowns*.

Goal: Solve the simultaneous equations.
That is, find values for the variables that satisfy both equations simultaneously.

Graphical Solution



Our Goal

Study *systems of linear equations* in detail and learn how to solve them in general.

We will:

- define a *system of linear equations*;
- represent such a system as a *matrix*;
- describe a general solution method (using the matrix representation).

Linear Equations or Not?

(a) $x_1 = 6 - 4x_2$

(b) $x_1(1 - x_2) = x_3$

(c) $\sqrt{2}x_1 + x_2 = \pi x_3$

(d) $\sqrt{2x_1} + x_2 = \pi x_3$

Section 1.1:
Systems of Linear Equations

Definition: A *linear equation* in the variables or unknowns $x_1, \ldots, x_n$ is one that can be written in the form

$$a_1x_1 + \cdots + a_nx_n = b$$

where $a_1, \ldots, a_n$ and b are real (or complex) numbers. The numbers $a_1, \ldots, a_n$ are called the *coefficients*.

Definition: A *system of linear equations* or a *linear system* is a collection of one or more linear equations in the same unknowns $x_1, \ldots, x_n$.

e.g.,

$$\begin{cases} x + y = 8 \\ 4x + 2y = 22 \end{cases} \qquad \begin{cases} x_2 + 4x_3 = -8 \\ -x_1 - x_4 = 0 \\ x_1 + 8x_4 + x_5 = -1 \end{cases}$$

Definition: A *solution* of a linear system is a list of numbers $(s_1, \ldots, s_n)$ that makes every equation of the system true when $s_1, \ldots, s_n$ are substituted for $x_1, \ldots, x_n$ respectively.

Definition: The *solution set* of a linear system is the set of all solutions.

e.g.,

$$\begin{cases} x + y = 8 \\ 4x + 2y = 22 \end{cases} \quad \text{has solution } (x, y) = (3, 5).$$

The solution set is the singleton set $\{(3, 5)\}$.

Fact: (proved later) Any system of linear equations has either:

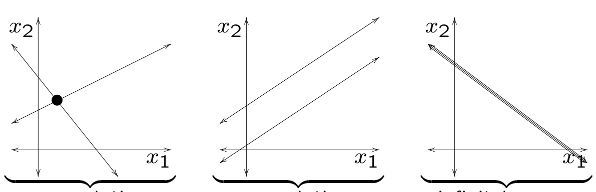
(a) No solutions;

(b) Exactly one (unique) solution;

(c) Infinitely many solutions.

No other result is possible.

Remember: (2 equations in 2 unknowns)



one solution — no solution — infinitely many related solutions

Definition: A linear system is *consistent* if it has at least one solution. Otherwise, it is *inconsistent*.

Matrix Notation

Definition: A *matrix* (plural: *matrices*) is a rectangular table (2-dimensional array) of numbers. Its *size* is the number of rows and columns. A matrix with m rows and n columns is an $m \times n$ *matrix*.

The numbers in a matrix are its *entries* or *elements*.

e.g.,

2×3 matrix $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 2 \end{bmatrix}$ $\quad$ 3×3 matrix $\begin{bmatrix} 3 & 0 & 1 \\ -1 & \frac{4}{5} & 2 \\ \sqrt{2} & 80 & 0 \end{bmatrix}$

1×1 matrix $\begin{bmatrix} -\pi \end{bmatrix}$ $\quad$ 4×1 matrix $\begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}$

Note: Rows and columns are counted starting with 1.

Given a linear system:

$$\begin{cases} \quad x_2 + 4x_3 \quad = -8 \\ -x_1 \quad - x_4 \quad = 0 \\ x_1 \quad + 8x_4 + x_5 = -1 \end{cases}$$

Definition: The *coefficient matrix* of a linear system is the matrix of coefficients:

$$\begin{bmatrix} 0 & 1 & 4 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 8 & 1 \end{bmatrix}$$

The *augmented matrix* also includes the right-hand side as an additional column:

$$\begin{bmatrix} 0 & 1 & 4 & 0 & 0 & -8 \\ -1 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 8 & 1 & -1 \end{bmatrix}$$

$$\begin{cases} x + y = 8 \\ 4x + 2y = 22 \end{cases} \qquad \begin{bmatrix} 1 & 1 & 8 \\ 4 & 2 & 22 \end{bmatrix}$$

$$\boxed{\begin{array}{r} 4x + 2y = 22 \\ -4(\quad x + y = 8\ ) \\ \hline -2y = -10 \end{array}} \qquad \boxed{R2 \to R2 - 4R1}$$

$$\begin{cases} x + y = 8 \\ -2y = -10 \end{cases} \qquad \begin{bmatrix} 1 & 1 & 8 \\ 0 & -2 & -10 \end{bmatrix}$$

$$\boxed{\begin{array}{c} -\frac{1}{2}(-2y = -10) \\ \hline y = 5 \end{array}} \qquad \boxed{R2 \to -\tfrac{1}{2}R2}$$

$$\begin{cases} x + y = 8 \\ y = 5 \end{cases} \qquad \begin{bmatrix} 1 & 1 & 8 \\ 0 & 1 & 5 \end{bmatrix}$$

$$\boxed{\begin{array}{r} x + y = 8 \\ -(\quad y = 5\ ) \\ \hline x \quad = 3 \end{array}} \qquad \boxed{R1 \to R1 - R2}$$

$$\begin{cases} x = 3 \\ y = 5 \end{cases} \qquad \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 5 \end{bmatrix}$$

Definition: *Elementary row operations (EROs)* on a matrix are:

1. (Replacement) Replace a row by the sum of itself and a multiple of another row;
2. (Interchange) Interchange two rows;
3. (Scaling) Multiply a whole row by a nonzero constant.

Fact: EROs are reversible. That is, for every ERO there is another ERO that "undoes" it:

original ERO	undone by
$R1 \to R1 + c \cdot R2$	$R1 \to R1 - c \cdot R2$
$R1 \leftrightarrow R2$	$R1 \leftrightarrow R2$
$R1 \to c \cdot R1$	$R1 \to \frac{1}{c} \cdot R1$

Definition: Two linear systems are *equivalent* if they have the same solution set.

Definition: Two matrices are *row equivalent* if there is a sequence of EROs that transforms the first to the second.

Fact: If the augmented matrices of two linear systems are row equivalent, the two systems are equivalent (have the same solution set).

$$\begin{tikzcd} \left\{\text{"Hard" linear system}\right. \arrow[rrr, Leftrightarrow, "\text{same solution sets}"] \arrow[d, Rightarrow] & & & \left\{\text{"Easy" linear system}\right. \\ \left[\text{Augmented matrix}\right] \arrow[r, "\text{ERO}"] & \cdots \arrow[rr, "\text{ERO}"] & & \left[\text{Augmented matrix}\right] \arrow[u, Rightarrow] \end{tikzcd}$$

Section 1.2: Row Reduction and Echelon Forms

Definition: A row of a matrix is a *zero row* if all its elements are zero. Otherwise, it is a *nonzero row*.

The *leading entry* of a nonzero row is the leftmost nonzero entry.

Definition: A matrix is in *echelon form* (or *row echelon form*) if all of the following hold:

1. all nonzero rows are above all zero rows;
2. each leading entry (of a nonzero row) is in a column to the right of the leading entry of the row above it;
3. all entries in a column below a leading entry are zero.

A matrix in echelon form is called an *echelon matrix*.

Examples:

in echelon form:

$$\begin{bmatrix} 0 & 2 & 1 & 1 & 3 \\ 0 & 0 & -1 & 0 & 8 \\ 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

not in echelon form:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 1 & 0 & 2 & -1 \\ 0 & 0 & -1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Definition: A matrix is in *reduced echelon form* (or *reduced row echelon form*) if it is in echelon form (satisfying 1–3 above) and **also**:

4. every leading entry is 1;
5. every leading entry is the *only* nonzero entry in its column (entries below and above are zero).

A matrix in reduced echelon form is called a *reduced echelon matrix*.

Examples:

in reduced echelon form:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 5 \end{bmatrix}$$

in echelon form but **not** reduced echelon form:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 & 4 & 3 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We can apply EROs to transform a matrix into echelon form or even reduced echelon form.

A matrix is equivalent to many different echelon matrices, but:

Theorem: (proved in a future class) Every matrix is row equivalent to one and only one reduced echelon matrix.

Definition: If a matrix A is row equivalent to a matrix U in echelon form, we call U an *echelon form of* A; if U is in reduced echelon form, we call U **the** *reduced echelon form of* A.

$$\begin{bmatrix} 1 & 1 & 8 \\ 4 & 2 & 22 \end{bmatrix} \xrightarrow{R2 \to R2 - 4R1} \begin{bmatrix} 1 & 1 & 8 \\ 0 & -2 & -10 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 8 \\ 4 & 2 & 22 \end{bmatrix} \xrightarrow[R2 \to R2 - R1]{\substack{R1 \leftrightarrow R2 \\ R2 \to 4R2}} \begin{bmatrix} 4 & 2 & 22 \\ 0 & 2 & 10 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 8 \\ 4 & 2 & 22 \end{bmatrix} \xrightarrow[R1 \to R1 - R2]{\substack{R2 \to R2 - 4R1 \\ R2 \to -\frac{1}{2}R2}} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 5 \end{bmatrix}$$

Fact: Every echelon form of a matrix has leading entries in the same positions.

Definition: We call these positions the *pivot positions* of the matrix. We call the columns in which pivot positions appear the *pivot columns*.